$\overline{A} \le \overline{B}$ means there is an injection $f: \overline{A} \to \overline{B}$.

<u>THM</u> If $\overline{A} \le \overline{B}$ and $\overline{B} \le \overline{A}$ then $\overline{A} = \overline{B}$. (anti-symmetry)

<u>Proof</u> Suppose $f: A \to B$, $g: B \to A$ are injections; we want to find an $h: A \to B$ which is a bijection.

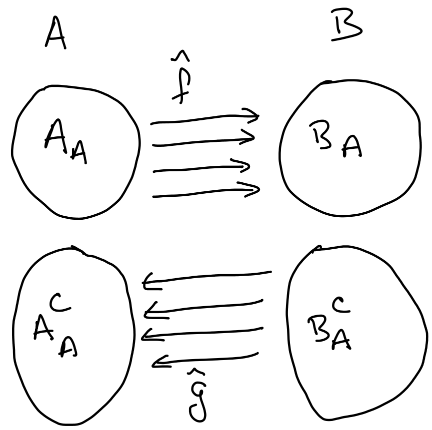
$a \in A$: $a = a_0, b_1, a_1, b_2, a_2, b_3, a_3 \cdots$

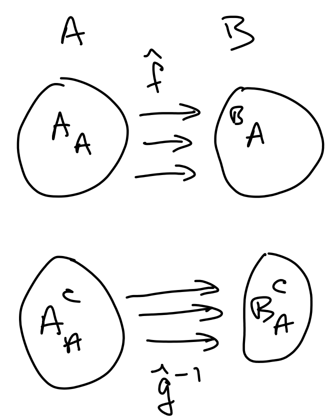
$b \in B$: $b = b_0, a_1, b_1, \cdots$

1) The sequence is infinite with distinct elements.
2) The sequence is finite: there is some b_k or a_k with no preimage.
3) The sequence repeats: $a_0, b_1, a_1 \cdots a_k = a_0$.

If $a = a_0, b_1, a_1 \cdots$, the elements form the lineage of a.

If $a = a_0 \cdots a_k \|$ then we say the lineage of a ends in A, or $a \in A_A$. Also if the lineage of b ends in A, we say $b \in B_A$.





$$h: A \to B, \quad h(a) = \begin{cases} \hat{f}(a) & \text{if } a \in A_A \\ \hat{g}^{-1}(a) & \text{if } a \in A_A^c \end{cases}$$

Suppose $f(a) = b$.

$a = a_0, b_1, a_1, b_2 \cdots$

$b, a_0, b_1, a_1, b_2 \cdots$

The lineage of a ends in A iff the lineage of b ends in A.

Define $\hat{f} : A_A \to B_A$ by $\hat{f}(a) = f(a)$.

$\hat{f}$ is a bijection:

1–1 : Suppose $\hat{f}(x) = \hat{f}(y)$, so $f(x) = f(y)$, so $x = y$ since f is 1–1.

onto : Pick $b \in B_A$: $b, a_1, b_1, a_2 \cdots a_k$ //

So $f(a_1) = b$, $a_1 \in A_A$. So $\hat{f}$ is onto B_A.

Suppose $g(b) = a$: $b \in B_A^c$ iff $a \in A_A^c$. (exercise)

Let $\hat{g}(b) = a$, ie, $\hat{g}(b) = g(b)$ for every $b \in B_A^c$.

So $\hat{g} : B_A^c \to A_A^c$.

Then $\hat{g}$ is a bijection from B_A^c to A_A^c. (exercise)

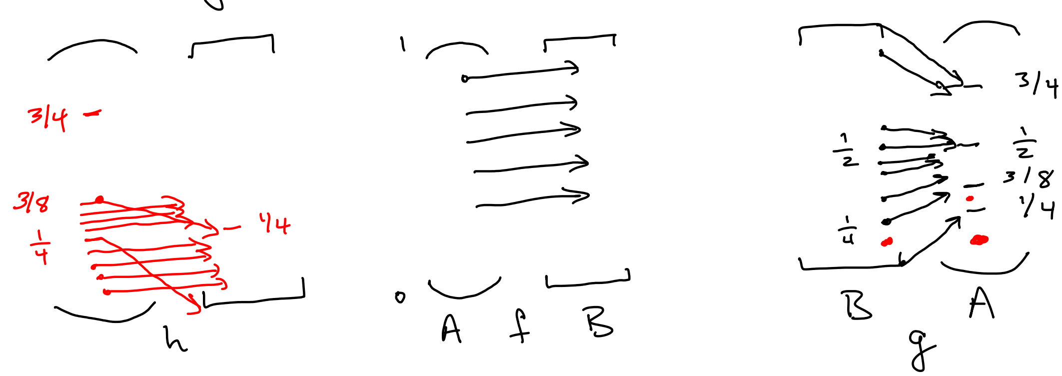
Then $h = \begin{cases} \hat{f}(a) & a \in A_A \\ \hat{g}^{-1}(a) & a \in A_A^c \end{cases}$ is a bijection from A to B.

$(0,1)$, $[0,1]$

A $\quad$ B

$f : A \to B$: $f(x) = x$

$g : B \to A$: $g(x) = \frac{x}{2} + \frac{1}{4}$



$(0, \frac{1}{4}) \subseteq A_A$

lineage of $\frac{1}{4}$: $\frac{1}{4}, 0$.

$\frac{1}{4} \notin A_A$.

$(\frac{1}{4}, \frac{3}{8}) \subseteq A_A$

$3/8 \notin A_A$

$\hat{g}^{-1}(3/8)$

$g(\frac{1}{4}) = \frac{1}{4}\frac{1}{2} + \frac{1}{4} = \frac{3}{8}$