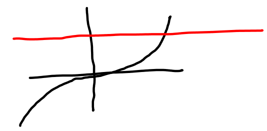
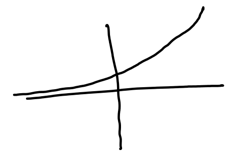
$f: A \to B$ is a bijection if f is both an injection and a surjection — it is 1-1 & onto. ~~A~~ bijection is also called a 1-1 correspondence.

$i_A : A \to A$, $i_A(a) = a$ for all $a \in A$.



$f(x) = x^3$, $f: \mathbb{R} \to \mathbb{R}$



~~$f(x) = e^x$, $f: \mathbb{R} \to \mathbb{R}$~~

$f(x) = e^x$, $f: \mathbb{R} \to \mathbb{R}^+$

f & g are inverses if $f: A \to B$ & $g: B \to A$, $f \circ g : B \to B$ is i_B, and $g \circ f : A \to A$ is i_A.

$f = x^3$, $g = x^{1/3}$

$(f \circ g)(x) = f(g(x)) = (x^{1/3})^3 = x$

$(g \circ f)(x) = (x^3)^{1/3} = x$.

<u>THM</u> f has an inverse iff f is a bijection.

<u>Proof</u> Suppose g is an inverse of f.

$g \circ f = \underset{\substack{\uparrow \\ \text{injective}}}{i_A}$, so by 4.4.1 f is injective.

Also by 4.4.1: Since $f \circ g = i_B$ is surjective, so is f.

Suppose f is a bijection. Let g be any pseudo inverse.

Thm. 4.5.2: Since f is injective, g is surjective, and since f is surjective, g is injective.

(So g is a bijection.)

$g(f(a))$ is some preimage of $f(a)$. Since f is 1-1, then $g(f(a)) = a$, $g \circ f = i_A$.

Suppose $b \in B$, $f(g(b))$: Since f is onto, $b \in$ range of f, so $g(b)$ is a preimage of b, so $f(g(b)) = b$.

So $f \circ g = i_B$.

<u>THM</u> If f has an inverse, the inverse is unique.

<u>Proof</u> Suppose g_1 & g_2 are inverses of f.

$$\begin{aligned}(g_1 \circ f) \circ g_2 &= i_A \circ g_2 = g_2 \\ \parallel \\ g_1 \circ (f \circ g_2) &= g_1 \circ i_B = g_1\end{aligned}$$

f has only one inverse so if f has an inverse we call it f^{-1}.

Even if f has no inverse, $f^{-1}(Y)$, $Y \subseteq B$, makes sense, but if $y \in B$, $f^{-1}(y)$ does not make sense.

If f has an inverse, then both make sense.

<u>THM</u> a) composition of 2 bijections is a bijection:
if f, g are bijections so is $f \circ g$.

b) The inverse of a bijection is a bijection.