Cardinality   Want to talk about sizes of sets.

If A is a finite set with 25 elements, its size or cardinality is 25.

When do we say two infinite sets A, B have the same size?

Def A & B have the same cardinality if there is a bijection $f: A \to B$.

If two finite sets have same size, say 25, there is a bijection from A to B:

$$A: a_1, a_2, \dots, a_{25}$$
$$B: b_1, b_2 \dots b_{25}$$

$$f(a_i) = b_i$$

If A is finite, $f: A \to B$ is a bijection, then B is finite & has the same size as A.

$$B = \{f(a_1), f(a_2) \dots f(a_n)\}$$

Infinite sets: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Do any of these have the same size? $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

$\mathbb{N} = \{1, 2, 3, 4 \dots\}$

If $\mathbb{N}$ & A have same cardinality, then A is countably infinite.

$$f: \mathbb{N} \to A, \quad A = \{f(1), f(2), f(3) \dots\}$$

A countable set is one that is finite or countably infinite.

If $\mathbb{N}$ & A have the same cardinality, we say $\mathbb{N} \approx A$.

A = set of positive even integers.

$$A = 2, 4, 6, 8 \dots$$
$$\uparrow \; \uparrow \; \uparrow \; \uparrow \; \dots$$
$$f(1) \; f(2) \; f(3) \; f(4)$$

$f: \mathbb{N} \to A$ by $f(n) = 2n$

onto: if x is even, $x = 2k$ for some k. $f(k) = 2k = x$.

1-1: Suppose $f(n) = f(m)$, $2n = 2m$, so $n = m$.

$\mathbb{N} \approx$ negative integers, $-1, -2, -3, -4 \ldots$

$f(n) = -n$, $f: \mathbb{N} \to \mathbb{Z}^-$

$\mathbb{N} \approx \mathbb{Z}$ $\qquad 0, 1, -1, 2, -2, 3, -3 \ldots$

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -\frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

$f(1) = -\frac{1-1}{2} = 0$

$f(2) = \frac{2}{2} = 1$

$f(3) = -\frac{3-1}{2} = -\frac{2}{2} = -1$

$f(4) = \frac{4}{2} = 2$

$f(5) = -\frac{5-1}{2} = -\frac{4}{2} = -2$

$\mathbb{N} \approx \mathbb{Q}^+$

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ \frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & \cdots \\ \frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} & \cdots \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} \\ \frac{1}{5} \end{array}$$

$1, 2, \frac{1}{2}, \frac{1}{3}, 3, 4, \frac{3}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, 5, 6, \ldots$

$$f(n) = \begin{cases} n/2 & n \text{ even} \\ -\frac{n-1}{2} & n \text{ odd} \end{cases}$$

$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 0, & 1, & -1, & 2, & -2, & 3, & -3, & 4, & -4, & 5, & -5 \end{array}$$

If p is a prime, $g(p^e) = p^{f(e+1)}$

$p = 3$ $\quad g(3^1) = 3^{f(1+1)} = 3^1$

$g(3^2) = 3^{f(2+1)} = 3^{f(3)} = 3^{-1} = 1/3$

$g(3^3) = 3^{f(4)} = 3^2 = 9$

$g(3^4) = 3^{f(5)} = 3^{-2} = 1/9$, 27, $1/27$, 81, $1/81$ $\ldots$

$n = p_1^{e_1} p_2^{e_2} p_3^{e_3} \cdots p_k^{e_k}$

$g(n) = g(p_1^{e_1}) g(p_2^{e_2}) \cdots g(p_k^{e_k})$

$= p_1^{f(e_1+1)} p_2^{f(e_2+1)} \cdots p_k^{f(e_k+1)}$

$n = 1$, $g(1) = 1$

$1 = 17^0$

$g(1) = g(17^0) = 17^{f(0+1)} = 17^0 = 1$

$$q = \frac{13^4 \cdot 17}{7^5 \cdot 11^2} = g(7^{10})\, g(11^4)\, g(13^7)\, g(17)$$

$$= \frac{p_1^{e_1} p_2^{e_2} \cdots}{q_1^{f_1} q_2^{f_2} \cdots}$$

<u>THM</u>

1) $A \approx A$

2) If $A \approx B$ then $B \approx A$

3) If $A \approx B$ and $B \approx C$ then $A \approx C$

Pf/ 2) Suppose $f: A \to B$ is a bijection; find a bijection $g: B \to A$

3) Suppose $f: A \to B$, $g: B \to C$ are bijections; find a bijection $h: A \to C$.