Relations: relationship between 2 objects.

$x \leq y \qquad x \mid y \qquad x = y$

$x \equiv y \pmod n \qquad x \subseteq y$

Transitivity: If $x \leq y \wedge y \leq z$ then $x \leq z$.

function: set of ordered pairs, $\{(a,b) \in A \times B \mid f(a) = b\}$

$\{(1,2),(2,4),(3,4)\} \quad A = \{1,2,3\}, \ B = \{1,2,\ldots 10\}$

Every element of A appears as the first element in exactly one ordered pair.

relation: any set of order pairs in $A \times B$.

Many relations are a subset of $A \times A$, a "relation on A".

(Not true: $x \in X$: $X \subseteq A$, $x \in A$.)

In the absence of a particular standard symbol we will use $\sim$ to represent a relation.

<u>Def</u> A relation on A is an equivalence relation if it has 3 properties:

1) $\forall a \in A \, (a \sim a)$ Reflexive
2) $\forall a, b \in A \, (a \sim b \Rightarrow b \sim a)$ Symmetric
3) $\forall a, b, c \in A \, (a \sim b \wedge b \sim c \Rightarrow a \sim c)$ Transitive

$x < y$: not reflexive.

If $\sim$ is an equiv. relation on A, $x \in A$,

$[x] = \{y \mid x \sim y\}$ is the equivalence class of x.

$[x]$: all integers congruent to x mod n: $n = 6, x = 2$

$[x] = \{\ldots, -4, 2, 8, 14 \ldots\}$

THM Suppose $\sim$ is an equivalence relation on A.
TFAE

1) $a \sim b$

2) $[a] \cap [b] \neq \emptyset$

3) $[a] = [b]$

Proof. $1 \Rightarrow 2$: Assume $a \sim b$, so $b \in [a]$ and $b \in [b]$.
So $b \in [a] \cap [b]$.

$2 \Rightarrow 3$: Prove $[a] \subseteq [b]$. Let $x \in [a] \cap [b]$.
Let $y \in [a]$.

$a \sim y$, $a \sim x$, so $y \sim a \wedge a \sim x$ so $y \sim x$.

Also: $b \sim x \wedge x \sim y$, so $b \sim y$, or $y \in [b]$.

$3 \Rightarrow 1$: Suppose $[a] = [b]$. Since $b \in [b]$, $b \in [a]$,
so $a \sim b$

If $\sim$ is an equivalence relation on A:

$A/_\sim$ = set of all equivalence classes.

If $\sim$ is "$\equiv \bmod (n)$", $n = 6$, $A = \mathbb{Z}$,

$$A/_\equiv = \{[0], [1], [2], \ldots [5]\} \text{ "=" } \mathbb{Z}_n.$$

We said: $[a] = [b]$ iff $a \equiv b \pmod n$.

Now: Theorem: If $[a] = [b]$ are equivalence classes, then
$[a] = [b] \Leftrightarrow a \equiv b$

$\mathbb{Z}_n$ is $\mathbb{Z}/_{\equiv_n}$.