Def $f: A \to B$. Define $x \sim y$ to mean $f(x) = f(y)$.

THM This is an equivalence relation

Pf/ Since $f(x) = f(x)$, $x \sim x$

Suppose $x \sim y$, so $f(x) = f(y)$, so $f(y) = f(x)$, so $y \sim x$.

If $x \sim y$, $y \sim z$, then $f(x) = f(y)$ and $f(y) = f(z)$,

so $f(x) = f(z)$, and $x \sim z$.

What is $[x]$? $[x] = \{y \mid x \sim y\} = \{y \mid f(x) = f(y)\}$

= set of all preimages of $f(x)$.

Suppose $R = f(A)$, range of f.

Define $\bar{f} : A/\sim \to R$ by $\bar{f}([x]) = f(x)$.

Claim $\bar{f}$ is a bijection.

well defined: Suppose $[x] = [y]$. $\bar{f}([x]) = f(x)$

$\bar{f}([y]) = f(y)$

Know: $x \sim y$ so $f(x) = f(y)$, so

$\bar{f}([x]) = \bar{f}([y])$

1-1: Suppose $\bar{f}([x]) = \bar{f}([y])$, so $f(x) = f(y)$, so

$x \sim y$, so $[x] = [y]$.

onto: Suppose $y \in R$. So $\exists x \in A\ (f(x) = y)$.

$\bar{f}(\underbrace{[x]}_{\in A/\sim}) = f(x) = y$.

Def $\pi : A \to A/\sim$ by $\pi(x) = [x]$. So

$\pi(x) = \pi(y)$ iff $[x] = [y]$ iff $x \sim y$.

π is a surjection: Given any $[x] \in A/\sim$, $\pi(x) = [x]$.

THM Every function $f: A \to B$ is a composition of an injection, a bijection, and a surjection: $\exists$ injection g, bijection h, and a surjection i such that $f = g \circ h \circ i$.

Proof: $A \xrightarrow[\pi]{} A/\sim \xrightarrow[\bar{f}]{} R \xrightarrow[g]{} B$

g is the inclusion function: $g(x) = x$.

$f = g \circ \bar{f} \circ \pi$:

$$(g \circ \bar{f} \circ \pi)(x) = (g \circ \bar{f})(\pi(x)) = (g \circ \bar{f})([x])$$

$$= g(\bar{f}([x])) = g(f(x)) = f(x)$$

"f has been factored"

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "\pi"'] & R \subseteq_{g} B \\ & A/\sim \arrow[u, "\bar{f}"'] \end{tikzcd}$$

"commutative diagram"

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr] & R \\ & A/\sim \arrow[u] \end{tikzcd}$$

"First isomorphism Theorem"