We can create new functions from old — composition, restriction

$f \circ g$, $f|_X$

If $\leq_A$ is a p.o. on A, and $S \subseteq A$, then $\leq$ induces an order on S, $\leq_S$:

$$\forall x y \in S \, (x \leq_S y \text{ iff } x \leq_A y)$$

<u>Lemma</u> If $\leq$ is a p.o. on A:

1) $\leq$ is p.o. on S
2) If $\leq$ is a total order on A, then $\leq$ is a total order on S.
3) If $\leq$ is a well-order on A, $\leq$ is a well-order on S.

$\mathbb{N}$ is ordered by "$|$" (divides)

If $S \subseteq \mathbb{N}$, S is also ordered by "$|$".

"S inherits these properties from A."

It is possible that $\leq$ on S has properties that $\leq$ on A does not.

$S = \{ 2^i \mid i \in \mathbb{N} \} \subseteq \mathbb{N}$ is ordered by "$|$".

$S: \; 2^1 \mid 2^2 \mid 2^3 \cdots$

Suppose $\leq_A$ is a p.o. on A, $\leq_B$ is a p.o. on B.

We define $\leq$ on $A \times B$ by:

$(a,b) \leq (c,d)$ if 1) $a < c$ or 2) $a = c \wedge b \leq d$. $\}$ always $\underline{a \leq c}$.

$A = \{1,2\}$, $B = \{3,4\}$: $(1,4) \leq (2,3)$

This is the lexicographic order.

<u>THM</u> If $\leq$ is the l.o. on $A \times B$ then

1) It is a p.o.
2) If $\leq_A$ & $\leq_B$ are total orders so is $\leq$.
3) If $\leq_A$ & $\leq_B$ are well-orders, so is $\leq$.

Proof 1) $(a,b) \le (a,b)$ — easy.

(R) Suppose $(a,b) \le (c,d) \wedge (c,d) \le (a,b)$. Then $a \le_A c \wedge c \le_A a$, so $a = c$.

(A-S) Then $b \le_B d \wedge d \le_B b$, so $b = d$. Thus $(a,b) = (c,d)$.

(T) if $(a,b) \le (c,d) \wedge (c,d) \le (e,f)$:

$a < c$ or $a = c \wedge b \le d$ $\qquad$ $c < e$ or $c = e \wedge d \le f$ $\quad \ldots$

2) Given (a,b) & (c,d), show either $(a,b) \le (c,d)$ or $(c,d) \le (a,b)$.

Either $a \le c$ or $c \le a$ ...

3) Suppose $S \subseteq A \times B$.

Let $T = \{ a \in A \mid \exists b \ (a,b) \in S \} \subseteq A$

There is a least $a_0 \in T$.

$U = \{ b \in B \mid (a_0, b) \in S \} \subseteq B$

There is a least $b_0 \in U$.

Claim: (a_0, b_0) is the least element of S.

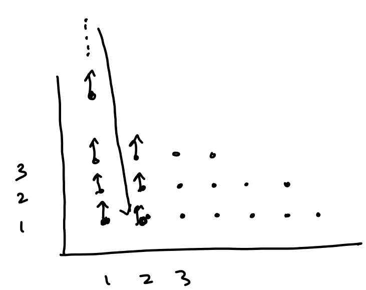
Suppose $(x,y) \in S$. Then $a_0 \le x$.

if $a_0 < x$, so $(a_0, b_0) \le (x,y)$.

otherwise, $a_0 = x$, so (a_0, b_0) & $(\underset{\overset{\shortparallel}{a_0}}{x}, y) \in U$, so

$b_0 \le y$, and $(a_0, b_0) \le (x,y)$.

$\mathbb{N} \times \mathbb{N}$



$\underbrace{(1,1) \le (1,2) \le (1,3) \ldots}$ $\le$ $\underbrace{(2,1) \le (2,2) \ldots}$