$\mathbb{N} \approx \mathbb{Q}$: as orders they are much different.

$\{2^i \mid i \in \mathbb{N}\} = \{2, 2^2, 2^3 \cdots\}$: this set has the same order as $\mathbb{N}$.

Suppose A & B are p.o. sets. ($\cancel{\leq_A, \leq_B}$ $\leq$)

We say A & B are isomorphic if there is a bijection $f: A \to B$, and $\forall x, y \in A$ ($x \leq y$ iff $f(x) \leq f(y)$).

"We can match the elements of A with those of B so that the order is preserved"

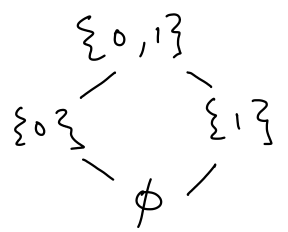
Rename	as
1	2^1
2	2^2
3	2^3
4	2^4
$\vdots$	$\vdots$

$\mathbb{N}, \mathbb{Q}$: there are bijections from $\mathbb{N}$ to $\mathbb{Q}$, say $f: \mathbb{N} \to \mathbb{Q}$.

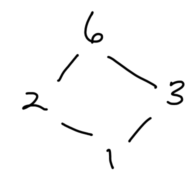
$f(1) = q \qquad f(x) = q - 1$

$1 \leq x \qquad q - 1 < q$

$\mathcal{P}(\{0,1\})$



$\{1, 2, 3, 6\}$, $|$



$f(\{0,1\}) = 6$, $f(\{0\}) = 2$, $f(\{1\}) = 3$, $f(\phi) = 1$

THM If A, B, C are p.o. then

1) $A \cong A$
2) If $A \cong B$ then $B \cong A$
3) If $A \cong B \wedge B \cong C$ then $A \cong C$

Proof (3) $f: A \to B$, $g: B \to C$. $g \circ f: A \to C$.

$g \circ f$ is a bijection. ✓

Suppose $x, y \in A$. Then $x \le y$ iff $f(x) \le f(y)$ iff $g(f(x)) \le g(f(y))$

so $x \le y$ iff $(g \circ f)(x) \le (g \circ f)(y)$

Example Let $I_a = (-\infty, a]$, $a \in \mathbb{R}$.

$S = \{ I_a \mid a \in \mathbb{R} \}$, ordered by $\subseteq$.

$\mathbb{R} \cong S$.

$f: \mathbb{R} \to S$, $f(a) = I_a$.

f is 1-1: Suppose $I_a = I_b$. $a \in I_a = I_b$, so $a \in I_b$, so $a \le b$
$b \in I_b = I_a$, so $b \in I_a$, so $b \le a$
so $a = b$.

$a \le b$ iff $I_a \subseteq I_b$

Suppose $a \le b$. Let $x \in I_a$, so $x \le a \le b$, so $x \in I_b$

Suppose $I_a \subseteq I_b$. $a \in I_a \subseteq I_b$, so $a \in I_b$, so $a \le b$.

If A is a p.o. then $I_a = \{ x \le a \mid a \in A \}$, $S = \{ I_a \mid a \in A \}$
ordered by $\subseteq$.

THM $\phi: A \to S$, $\phi(a) = I_a$ is an isomorphism, so
$A \cong S$.

✓

Every partial order is isomorphic to some collection of sets
with $\subseteq$ as the order.