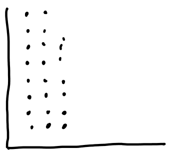
All p.o. are "suborders" of some power set with "$\subseteq$" as the order relation.

Theorem If A is a countable, totally ordered set then it is isomorphic to a subset of $\mathbb{Q}$.

Example Lexicographic order on $\mathbb{N} \times \mathbb{N}$.



$$(1,1) < (1,2) < (1,3) \cdots \quad (2,1) < (2,2) < (2,3) \cdots$$

$$0, \tfrac{1}{2}, \tfrac{3}{4}, \tfrac{7}{8}, \tfrac{15}{16}, \cdots \quad 1, \tfrac{3}{2}, \tfrac{7}{4}, \cdots \quad 2, \tfrac{5}{2}, \cdots$$

Proof Since A is countable, we can list A in a sequence: $a_1, a_2, a_3, \ldots$.

Define $f : A \to \mathbb{Q}$, one a_i at a time.

Let $f(a_1) = 0$.

Suppose we have defined $f(a_1) \ldots f(a_n)$ so that

$\forall x, y \in \{a_1 \ldots a_n\}$ ($x \le y$ iff $f(x) \le f(y)$).

Partition $\{a_1 \ldots a_n\}$ into

$$X = \{a_i \mid a_i \le a_{n+1}\}$$

$$Y = \{a_i \mid a_{n+1} \le a_i\}$$

$$f(X) = \{f(a_i) \mid a_i \in X\}$$

$$f(Y) = \{f(a_i) \mid a_i \in Y\}$$

Every element of $f(X)$ is less than every element of $f(Y)$

if $q \in f(X)$, $q = f(x)$, $x \in X$, $x \le a_{n+1}$.

if $r \in f(Y)$, $r = f(y)$, $y \in Y$, $y \ge a_{n+1}$.

So $x \le y$, so $f(x) \le f(y)$.

$f(X)$ & $f(Y)$ are finite subsets of $\mathbb{Q}$. There is a maximum M in $f(X)$, and a minimum m in $f(Y)$, $M < m$.

Let $q = \frac{M+m}{2}$. Let $f(a_{n+1}) = q$.

Claim: $\forall x, y \in \{a_1 \cdots a_{n+1}\}\ (x \leq y \text{ iff } f(x) \leq f(y))$.

Case 1. $x, y \in \{a_1 \cdots a_n\}$. True by assumption.

Case 2. $x = a_{n+1}$, $y = a_i$.

If $x \leq y$, $a_{n+1} \leq a_i$, then $a_i \in Y$. So $f(a_i) \in f(Y)$,

so $f(a_i) > q = f(x)$, ie, $f(y) \geq f(x)$.

If $f(x) = q \leq f(y) = f(a_i)$, then $a_i \in f(Y)$.

So $a_i \in Y$, so $a_i \geq a_{n+1}$ or $y \geq x$.

Eventually, we have defined $f(x)$ for every $x \in A$.

Then: $x \leq y$ iff $f(x) \leq f(y)$ for every $x, y \in A$.

Pick any $x, y \in A$, $x = a_i$, $y = a_j$. For some n,

$x, y \in \{a_1 \cdots a_n\}$, eg, $n = \max(i, j)$.

So $x \leq y$ iff $f(x) \leq f(y)$.