Constructing the real numbers.

Starting with the integers, $\mathbb{Z}$, we want to construct the real numbers, showing that there really is a set that acts the way the real numbers should act.

First, $\mathbb{Q}$. Consider $A = \{(a,b) \mid a,b \in \mathbb{Z}, b \neq 0\}$ "(a,b) is $\frac{a}{b}$"

$\frac{1}{2} = \frac{2}{4}$ as a rational number, but $(1,2) \neq (2,4)$.

$\frac{a}{b} = \frac{c}{d}$ iff $ad = bc$. $(a,b) \sim (c,d)$ iff $ad = bc$

"$\sim$" is an equivalence relation:

1) $(a,b) \sim (a,b)$ because $ab = ba$
2) Suppose $(a,b) \sim (c,d)$: $ad = bc$] same statement.
   $(c,d) \sim (a,b)$: $cb = da$]
3) Suppose $(a,b) \sim (c,d) \sim (e,f)$

$$ad = bc \qquad cf = de$$
$$adf = bcf \qquad bcf = bde$$
So $adf = bde$

so $af = be$, so $(a,b) \sim (e,f)$

A rational number is $[(a,b)]$, or if $\mathbb{Q} = A/\sim$, then this is the familiar rational numbers. Let's call $[(a,b)] = \frac{a}{b}$.

<u>Define</u> $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$. Need: If $\frac{a}{b} = \frac{a'}{b'}$, $\frac{c}{d} = \frac{c'}{d'}$ then

$\frac{a}{b} + \frac{c}{d} = \frac{a'}{b'} + \frac{c'}{d'}$. Know: $ab' = a'b$ and $cd' = dc'$.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}, \quad \frac{a'}{b'} + \frac{c'}{d'} = \frac{a'd'+b'c'}{b'd'}.$$

Need: $b'd'(ad+bc) = bd(a'd'+b'c')$

$$\underbrace{b'd'a}d + b'\underbrace{d'bc} = bda'd' + bdb'c'$$

$$\underline{bd'a'd} + \underline{b'dbc'} = \underline{bda'd'} + \underline{bdb'c'} \quad \checkmark$$

$$\frac{a}{b} < \frac{c}{d} \quad \text{iff} \left\{ \begin{array}{ll} ad < bc & \text{if } bd > 0 \\ ad > bc & \text{if } bd < 0 \end{array} \right\} \quad \text{iff} \quad [(a,b)] < [(c,d)]$$

If $\frac{a}{b} = \frac{a'}{b'} \wedge \frac{c}{d} = \frac{c'}{d'}$ then $\frac{a}{b} < \frac{c}{d}$ iff $\frac{a'}{b'} < \frac{c'}{d'}$.

$$\boxed{ab' = ba' \qquad cd' = dc'}$$

Suppose $bd > 0$, $b'd' > 0$.

$\frac{a}{b} < \frac{c}{d}$ iff $ad < bc$, $\frac{a'}{b'} < \frac{c'}{d'}$ iff $a'd' < b'c'$.

Need to show: $\underline{ad < bc}$ iff $\underline{a'd' < b'c'}$.

$ad < bc$

$\underline{adb'}$? bcb' $\qquad$? = "<" if $b' > 0$, ">" if $b' < 0$.

$a'db$? bcb'

$a'dbd'$?? $bcb'd'$ $\qquad$?? = ? if $d' > 0$, opposite if $d' < 0$

$a'dbd' < bc\underline{b'd'}$

$a'dbd' < bc'b'd$ $\qquad$ $bd > 0$

$a'd' < c'b'$

$\mathbb{Q}$ is now known. Build the real numbers: fill in the gaps.

"Dedekind cut"

Suppose $A \subseteq \mathbb{Q}$: 1) $A \neq \emptyset$

2) $A \neq \mathbb{Q}$

3) $\forall x\,(x \in A \Rightarrow \forall y\,(y < x \Rightarrow y \in A))$ "A is downward closed"

4) A has no largest element.

Roughly: such an A is $(-\infty, r)$.
$\qquad\qquad\qquad\qquad\qquad\qquad\quad \uparrow$ $\mathbb{R}$.

Examples: $\underline{\{x \mid x < 1\}}$, $\{x \mid x < -2\}$ ....

Define $\mathbb{R}$ = set of all such Dedekind cuts.

The "cut" is A, A^c. If $A = (-\infty, 1)$, $A^c = [1, \infty)$

$(-\infty, \sqrt{2})^c = (\sqrt{2}, \infty)$

Define an isomorphic copy of $\mathbb{Q}$: $(-\infty, q)$ is the rational number q.

Suppose A, B are in $\mathbb{R}$, that is, are Dedekind cuts.

$A+B = \{a+b \mid a \in A, b \in B\}$. Want $A+B = (-\infty, \frac{11}{2})$

$A = \{x \mid x < \frac{1}{2}\} = (-\infty, \frac{1}{2})$ "$= \frac{1}{2}$"

$B = (-\infty, 5)$ "$= 5$"

If $a \in A, b \in B$, $a+b < \frac{1}{2} + 5 = \frac{11}{2}$.

Need: If $q < \frac{11}{2}$, $q \in A+B$.

If $q < 5$: $q = 0 + q \in A+B$

If $q \geq 5$, $q < \frac{11}{2}$: $q = \left(\frac{1}{2} - \frac{11/2 - q}{2}\right) + \left(5 - \frac{11/2 - q}{2}\right)$

$= \frac{1}{2} + 5 - \cancel{2}\,\frac{11/2 - q}{\cancel{2}} = \frac{11}{2} - \left(\frac{11}{2} - q\right) = q$

$\frac{\frac{11}{2} - q}{2} > 0$: $\frac{1}{2} - \frac{11/2 - q}{2} < \frac{1}{2}$, $5 - \frac{11/2 - q}{2} < 5$

In general: $A+B$ is a Dedekind cut.

$\underline{A \cdot B} = \cancel{\{ab \mid a \in A, b \in B\}}$

$A = \frac{1}{2}$, $B = 5$

$AB = \frac{5}{2} = (-\infty, \frac{5}{2})$? But $-1 \in A$, $-5 \in B$, $(-1)(-5) = 5 > \frac{5}{2}$

If $A \geq 0, B \geq 0$

$A \cdot B = \{ab \mid a \in A \wedge a \geq 0 \wedge b \in B \wedge b \geq 0\} \cup \{x \mid x < 0\}$

[$A < B$ iff $A \subseteq B$. Examples: "$\frac{1}{2} < 5$" because $(-\infty, \frac{1}{2}) \subseteq (-\infty, 5)$]

Example: $\frac{1}{2} \cdot 5 = (-\infty, \frac{1}{2}) \cdot (-\infty, 5) = \{ab \mid a, b \geq 0, a \in A, b \in B\} \cup \{x \mid x < 0\}$

$= (-\infty, 5/2)$ ✓

If $a < \frac{1}{2}$, $b < 5$, then $ab < \frac{5}{2}$.

Also need: If $q < \frac{5}{2}$ then $q \in AB$.

$2q < 5$ because $q < \frac{5}{2}$. Pick $b \in (2q, 5)$.

Let $a = \frac{q}{b} < \frac{q}{2q} = \frac{1}{2}$. $a \in (-\infty, \frac{1}{2})$, $b \in (-\infty, 5)$, $ab \in AB$

$ab = q$.

Define $-A$ to be $\{c - a \mid c < 0 \wedge a \in A^c\}$.

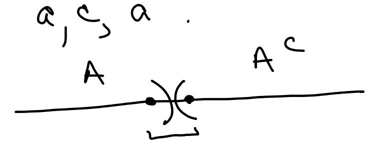
$-A$ is a cut.

$$A + -A = \{a + b \mid a \in A, b \in -A\}$$

$$= \{a + c - \bar{a} \mid a \in A, c < 0, \bar{a} \in A^c\} \overset{?}{=} \{x \mid x < 0\}$$

$\bar{a} > a$, so $a - \bar{a} < 0$, so $a + c - \bar{a} < 0$.

Suppose $q < 0$. Need $q = a + c - \bar{a}$ for some $a, c, \bar{a}$.



Pick a & $\bar{a}$ so that $q < a - \bar{a} < 0$.

so $q - a + \bar{a} < 0$.

$q = a + (q - a + \bar{a}) - \bar{a}$, so $q \in A + -A$.

If A & B are any numbers,

$$AB = -(-A)B$$

$$= (-A)(-B) \quad ?$$

$$= -A(-B)$$

If $\begin{cases} A < 0 \\ B > 0 \end{cases}$ define $AB = -\underline{(-A)B}$

If $\begin{cases} A < 0 \\ B < 0 \end{cases}$ define $AB = (-A)(-B)$

If $\begin{cases} A > 0 \\ B < 0 \end{cases}$ define $AB = -A(-B)$

$$\text{"}-2\text{"} \cdot \text{"}5\text{"} = \overset{A}{\{x \mid x < -2\}} \cdot \overset{B}{\{x \mid x < 5\}} = -(-A)(B)$$

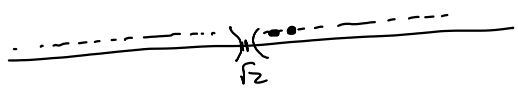
$$= -\{x \mid x < 2\} \cdot \{x \mid x < 5\} = -\{x \mid x < 10\} = \{x \mid x < -10\}$$

Suppose $X \subseteq \mathbb{R}$. An upper bound for the set X is an $M \in \mathbb{R}$ such that $\forall r \in X \, (r \leq M)$. If there is an upper bound, X is bounded above.

The least upper bound of X is the smallest M that is an upper bound.

Property of $\mathbb{R}$: Every X which is bounded above has a least upper bound.

Not true in $\mathbb{Q}$: $(-\infty, \sqrt{2}) \subseteq \mathbb{Q}$. This set is bounded above, because 2 is an upper bound. But the least upper bound is $\sqrt{2}$ in $\mathbb{R}$, and $\sqrt{2} \notin \mathbb{Q}$.



X is bounded above; each $r \in X$ is a cut.

$M = \bigcup_{r \in X} r$ is a cut and because $r \subseteq M$, M is an upper bound, in fact a least upper bound.

Suppose $a < M$ & a is an upper bound for X:

$\forall r \in X \, (r \leq a < M)$. Get a contradiction:

$\bigcup_{r \in X} r \neq M$.

Is $\sqrt{2} \in \mathbb{R}$.

Let $A = \{x \mid x^2 < 2\}$.

1. Show A is a cut, and $A > 0$.
2. Show $A \cdot A = 2$.

A cut A must have no greatest element.

If $a \in A$ need a $y > a$ and $y \in A$.

Let $y = \dfrac{2a+2}{a+2}$ show: $a < y$ and $y \in A$ ($y^2 < 2$).