On the Geographical Problem of the Four Colours.

By A. B. Kempe, B. A., *London, England.*

If we examine any ordinary map, we shall find in general a number of lines dividing it into districts, and a number of others denoting rivers, roads, etc. It frequently happens that the multiplicity of the latter lines renders it extremely difficult to distinguish the boundary lines from them. In cases where it is important that the distinction should be clearly marked, the artifice has been adopted by map-makers of painting the districts in different colours, so that the boundaries are clearly defined as the places where one colour ends and another begins; thus rendering it possible to omit the boundary lines altogether. If this clearness of definition be the sole object in view, it is obviously unnecessary that non-adjacent districts should be painted different colours; and further, none of the clearness will be lost, and the boundary lines can equally well be omitted, if districts which merely meet at one or two points be painted the same colour. (See Fig 1.)

This method of definition may of course be applied to the case of any surface which is divided into districts. I shall, however, confine my investigations primarily to the case of what are known as simply or singly connected surfaces, *i. e.* surfaces such as a plane or sphere, which are divided into two parts by a circuit, only referring incidentally to other cases.

If, then, we take a simply connected surface divided in any manner into districts, and proceed to colour these districts so that no two adjacent districts shall be of the same colour, and if we go to work at random, first colouring as many districts as we can with one colour and then proceeding to another colour, we shall find that we require a good many different colours; but, by the use of a little care, the number may be reduced. Now, it has been stated somewhere by Professor De Morgan that it has long been known to map-makers as a matter of experience—an experience however probably confined to comparatively simple cases—that *four* colours will suffice in any case. That four colours may be necessary will be at once obvious on consideration of the case of one district surrounded by three others, (see Fig. 2), but that four colours will suffice in all cases is a fact which is by no means obvious, and has rested hitherto, as far as I know, on the experience I have mentioned, and on the statement of Professor De Morgan, that the fact was no doubt true. Whether that statement was one merely of belief, or whether Professor De Morgan, or any one else, ever gave a proof of it, or a way of colouring any given map, is, I believe, unknown; at all events, no answer has been given to the query to that effect put by Professor Cayley to the London Mathematical Society on June 13th, 1878, and subsequently, in a short communication to the Proceedings of the Royal Geographical Society, Vol. I, p. 259, Professor Cayley, while indicating wherein the difficulty of the question consisted, states that he had not then obtained a solution. Some inkling of the nature of the difficulty of the question, unless its weak point be discovered and attacked, may be derived from the fact that a very small alteration in one part of a map may render it necessary to recolour it throughout. After a somewhat arduous search, I have succeeded, suddenly, as might be expected, in hitting upon the weak point, which proved an easy one to attack. The result is, that the experience of the map-makers has not deceived them, the maps they had to deal with, viz: those drawn on simply connected surfaces, can, in every case, be painted with four colours. How this can be done I will endeavour—at the request of the Editor-in-Chief—to explain.

Suppose that we have the surface divided into districts in any way which admits of the districts being coloured with four colours, viz: blue, yellow, red, and green; and suppose that the districts are so coloured. Now if we direct our attention to those districts which are coloured red and green, we shall find that they form one or more detached regions, *i. e.* regions which have no boundary in common, though possibly they may meet at a point or points. These regions will be surrounded by and surround other regions composed of blue and yellow districts, the two sets of regions making up the whole surface. It will readily be seen that we can interchange the colours of the districts in one or more of the red and green regions without doing so in any others, and the map will still be properly coloured. The same remarks apply to the regions composed of districts of any other pair of colours. Now if a region composed of districts of any pair of colours, say red and green as before, be of either of the forms shown in Figures 3 and 4, it will separate the surface into two parts, so that we may be quite certain that no yellow or blue districts in one part can belong to the same yellow and blue region as any yellow or blue district in the other part. Thus any specified blue district, for example, in one part can, by an interchange of the colours in the yellow and blue region to which it belongs, be converted into a yellow district, whilst any specified yellow district in the other part remains yellow.

Now let us consider the state of things at a point where three or more boundaries and districts meet. It will be convenient to term such a point a *point of concourse.* If three districts meet at the point, they must be coloured with three different colours. If four, they may be coloured with two or three colours only in some cases, but on the other hand they may be coloured with four, as in Fig. 5. If the districts a and c in this case belong to different red and green regions, we can interchange the colours of the districts in one of these regions, and the result will be that the districts a and c will be of the same colour, both red or both green. If a and c belong to the same red and green region, that region will form a ring as in Fig. 4, and b will be in one of the parts into which it divides the surface and d in the other, so that the yellow and blue region to which b belongs, will be different from that to which d belongs; if, therefore, we interchange the colours in either of these regions, b and d will be of the same colour, both yellow or both blue. Thus we can always reduce the number of colours which meet at the point of concourse of four boundaries to three.

The same thing may be shown in the case of points of concourse where five boundaries meet. The districts meeting at the point may happen to be coloured with only three colours, but they may happen to be coloured with four. Fig. 6 shows the only form which the colouring can take in that case, one colour of course occurring twice. If a and c belong to different yellow and red regions, interchanging the colours in either, a and c become both yellow or both red. If a and c belong to the same yellow and red region, see if a and d belong to different green and red regions; if they do, interchanging the colours in either region, a and d become both green or both red. If a and c belong to the same yellow and red region, and a and d belong to the same green and red region, the two regions cut off b from e, so that the blue and green region to which b belongs is different from that to which d and e belong, and the blue and yellow region to which e belongs is different from that to which b and c belong. Thus, interchanging the colours in the blue and green region to which b belongs, and in the blue and yellow region to which e belongs, b becomes green and e yellow, a, c and d remaining unchanged. In each of the three cases the number of colours at the point of concourse is reduced to three.

It will be unnecessary for my purpose to take the case of a larger number of boundaries. Later on, we shall see that we can arrange the colours so that not only will three colours only meet at any given point of concourse, however many boundaries meet there, but also at no point of concourse in the map will four colours appear. It is, however, at present, enough, (and I have proved no more), that if less than six boundaries meet at a point we can always rearrange the colours of the districts so that the number of colours at that point shall only be three.

Before leaving this part of the investigation, I may point out that it does not apply to the case of other surfaces. A glance at Fig 7, which represents an anchor ring, will show that a ring-shaped district, a a, if it *clasps* the surface, does not divide it into two parts, so that the foregoing proof fails. In fact, six colours may be required to colour an anchor ring. For, if two *clasping* boundaries be described so as to divide the ring into two bent cylindrical portions, and if each portion be divided into three parts by longitudinal boundaries, a, b, c being the three parts on one and d, e, f being those on the other, so that a abuts on d and e at one end, and on e and f at the other; b abuts on e and f at one end, and on f and d at the other; c abuts on f and d at one end, and on d and e at the other, then a, b, c, d, e, f must all be of different colours.

Returning to the case of the simply connected surface, and putting aside for the moment the question of colouring, let us consider some points as to the structure of the map on its surface. This map can have in it *island-districts* having one boundary (Fig. 8); and *island-regions* (Fig. 9) composed of a number of districts; also, *peninsula-districts*, having one boundary and one point of concourse (Fig. 10); and *peninsula-regions* (Fig. 11); *complex-districts*, which have islands and peninsulas in them; and *simple-districts* which have none, and have as many boundaries as points of concourse (Fig. 12). It should also be noticed that, with the exception of those boundaries which are endless, such as that in Fig. 8, and those which have one point of concourse such as that in Fig. 10, every boundary ends in two points of concourse; and further, that every boundary belongs to two districts.

Now, take a piece of paper and cut it out to the same shape as any simple- island- or peninsula-district, but rather larger, so as just to overlap the boundaries when laid on the district. Fasten this *patch* (as I shall term it) to the surface and produce all the boundaries which meet the patch, (if there be any, which will always happen except in the case of an island), to meet at a point, (a point of concourse) within the patch. If only two boundaries meet the patch, which will happen if the district be a peninsula, join them across the patch, no point of concourse being necessary. The map will then have one district less, and the numbers of boundaries will also be reduced. Fig. 13 shows the district before the patch is put on, the place where it is going to be, being indicated by the dotted line, and Fig. 14 shows what is seen after the patch (again denoted by the dotted line) has been put on, and the boundaries have been produced to meet in a point on it. This patching process can be repeated as long as there is a simple district left to operate upon, the patches being in some cases stuck partially over others. If we confine our operations to an island or peninsula, we shall at length get rid of the island or peninsula, and doing this in the case of all the islands and peninsulas, complex-districts will be reduced to simple ones, and can be got rid of by the same process. We can thus, by continually patching, at length get rid of every district on the surface, which will be reduced to a single district devoid of boundaries or points of concourse. The whole map is patched out.

Now, reverse the process, and strip off the patches in the reverse order, taking off first that which was put on last, as each patch is stripped off it discloses a new district, and the map is *developed* by degrees.

Suppose that at any stage of this development, when we have stripped off a number of patches, there are on the surface

D districts
B boundaries
P points of concourse,

and suppose that after the next patch is stripped off there are

D' districts
B' boundaries
P' points of concourse.

If the patch has no point of concourse on it or line, *i. e.*, if when it is stripped off an island is disclosed,

$$P' = P$$
$$D' = D + 1$$
$$B' = B + 1.$$

If the patch has no point of concourse but only a single line, so that when it is stripped off a peninsula is disclosed,

$$P' = P + 1$$
$$D' = D + 1$$
$$B' = B + 2.$$

If the patch has a point of concourse on it where σ boundaries meet

$$P' = P + \sigma - 1$$
$$D' = D + 1$$
$$B' = B + \sigma.$$

In each case therefore

$$P' + D' - B' - 1 = P + D - B - 1,$$

i. e., at every stage of the development

$$P + D - B - 1$$

has the same value. But at the first stage

$$P = 0$$
$$D = 1$$
$$B = 0.$$

Therefore we always have

$$P + D - B - 1 = 0^{*}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

That is *in every map drawn on a simply connected surface the number of points of concourse and number of districts are together one greater than the number of boundaries*†.

Let d_1, d_2, d_3, etc., denote the number of districts at any stage, which have one, two, three, etc., boundaries, so that

$$D = d_1 + d_2 + d_3 + \ldots,$$

and let p_3, p_4, etc., denote the number of points of concourse, at the same stage of the development, at which three, four, etc., boundaries meet, so that

$$P = p_3 + p_4 + \ldots.$$

Then, since every boundary belongs to two districts,

$$2B = d_1 + 2d_2 + 3d_3 + \ldots,$$

and since every boundary ends in two points of concourse, except in the case of continuous boundaries which have no points of concourse, of which let there be β_0, and boundaries round peninsula districts which have one point of concourse, of which let there be β_1, therefore,

$$2B = 2\beta_0 + \beta_1 + 3p_3 + 4p_4 + \ldots.$$

Thus, since (1) may be written

$$(6D - 2B) + (6P - 4B) - 6 = 0,$$

we have $$5d_1 + 4d_2 + 3d_3 + 2d_4 + d_5 - \text{etc.} = 0,$$

the first five terms being the only positive ones. At least one, therefore, of the quantities $d_1 . d_2 . d_3 . d_4 . d_5$ must not vanish, *i. e. every map drawn on a simply connected surface must have a district with less than six boundaries.*

* The formula (1) was first stated as connecting the number of angular points, faces, and edges of a polyhedron by Cauchy.

† See note following this paper.

It may readily be seen that this proof applies equally well to an *island-region* or *peninsula-region* as to the whole map. The result is, that we can patch out any simply-connected map, never putting a patch on a district with more than five boundaries. Consequently, if we develop a map so patched out, since each patch, when taken off, discloses a district with less than six boundaries, not more than five boundaries meet at the point of concourse on the patch*. Of course districts which, when first disclosed, have only five boundaries may ultimately have thousands.

Returning to the question of colour, if the map at any stage of its development, can be coloured with four colours, we can arrange the colours so that, at the point of concourse on the patch next to be taken off, where less than six boundaries meet, only three colours shall appear, and, therefore, when the patch is stripped off, only three colours surround the disclosed district, which can, therefore, be coloured with the fourth colour, *i. e.* the map can be coloured at the next stage. But, at the first stage, one colour suffices, therefore, four suffice at all stages, and therefore, at the last. This proves the theorem and shows how the map may be coloured.

I stated early in the article that I should show that the colours could be so arranged that only three should appear at every point of concourse. This may readily be shown thus: Stick a small circular patch, with a boundary drawn round its edge, on every point of concourse, forming new districts. Colour this map. Only three colours can surround any district, and therefore the circular patches. Take off the patches and colour the uncovered parts the same colour as the rest of their districts. Only three colours surrounded the patches, and therefore only three will meet at the points of concourse they covered.

A practical way of colouring any map is this, which requires no patches. Number the districts in succession, always numbering a district which has less than six boundaries, not including those boundaries which have a district already numbered on the other side of them. When the whole map is numbered, beginning from the highest number, letter the districts in succession with four letters, $a.b.c.d$, rearranging the letters whenever a district has four round it, so that it may have only three, leaving one to letter the district with. When the whole map is lettered, colour the districts, using different colours for districts lettered differently.

*See note following this paper.

Two special cases should be noticed.

(1). If, excluding island and peninsula districts from the computation, every district is in contact with an even number of others along every circuit formed by its boundaries, three colours will suffice to colour the map.

(2). If an even number of boundaries meet at every point of concourse, two colours will suffice. This species of map is that which is made by drawing any number of continuous lines crossing each other and themselves any number of times.

If we lay a sheet of tracing paper over a map and mark a point on it over each district and connect the points corresponding to districts which have a common boundary, we have on the tracing paper a diagram of a "linkage," and we have as the exact analogue of the question we have been considering, that of lettering the points in the linkage with as few letters as possible, so that no two directly connected points shall be lettered with the same letter. Following this up, we may ask what are the linkages which can be similarly lettered with not less than n letters?

The classification of linkages according to the value of n is one of considerable importance. I shall not, however, enter here upon this question, as it is one which I propose to consider as a part of an investigation upon which I am engaged as to the general theory of linkages. It is for this reason also that I have preferred to treat the question discussed in this paper in the manner I have done, instead of dealing with the analogous linkage.

I will conclude with a theorem which can readily be obtained as a corollary to the preceding results. It is one of which I long endeavoured to obtain an independent proof, as a means of solving the four-colour problem. The polyhedra mentioned are to be understood to be simply connected ones. The theorem is this:

"*Polyhedra can be added to the faces of any polyhedron so that in the resulting polyhedron* (1) *the faces are all triangles*, (2) *the number of edges meeting at every angular point is a multiple of three.*

JUNE *23d, 1879.*

Fig. 1.

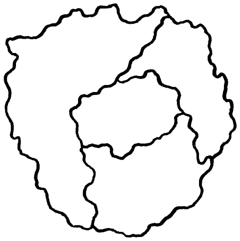

Fig. 2.

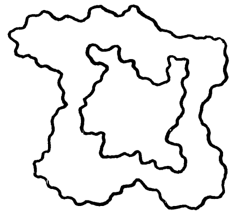

Fig. 3.

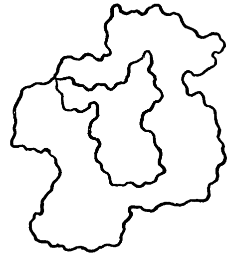

Fig. 4.

Fig. 5.

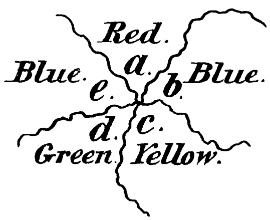

Fig. 6.

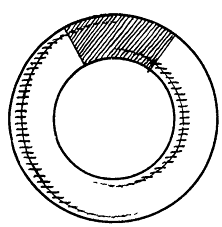

Fig. 7.

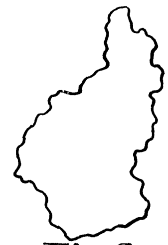

Fig. 8.

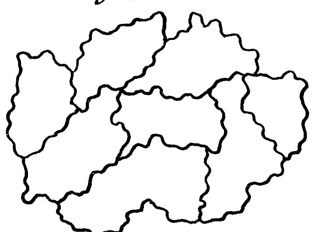

Fig. 9.

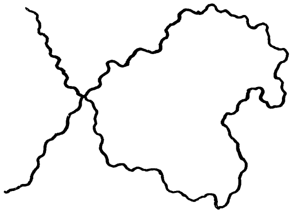

Fig. 10.

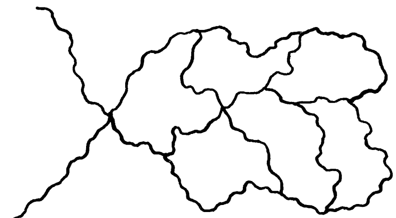

Fig. 11.

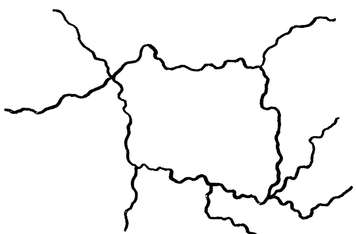

Fig. 12.

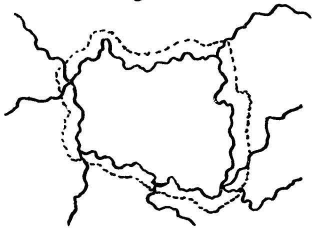

Fig. 13.

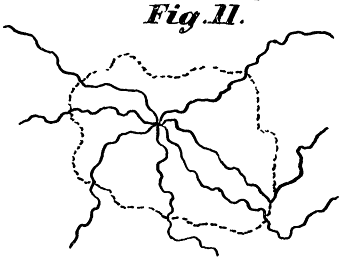

Fig. 14.

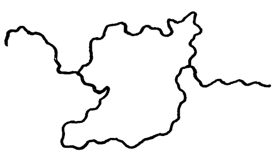

Fig. 15.

Fig. 16.

KEMPE, Geographical Problem.

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