Find the mistake!

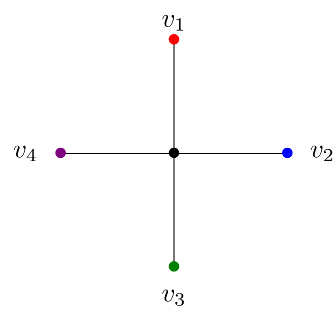
Theorem. *Every planar graph can be colored with 4 colors.*

Proof.

The proof is by induction on the number of vertices n; when $n \le 4$ this is trivial.

Now suppose G is planar on more than 4 vertices; we know some vertex v has degree at most 5. By the induction hypothesis, $G - v$ can be colored with 4 colors. Color the vertices of G, other than v, as they are colored in a 4-coloring of $G - v$. If $\mathrm{d}(v) \le 3$, then v can be colored with one of the 4 colors to give a proper coloring of G with 4 colors.

Now suppose $\mathrm{d}(v) = 4$. If the four neighbors of v are colored with three or fewer of the colors, then again v can be colored to give a proper coloring of G with 4 colors. So suppose the four neighbors are colored with 4 colors as shown:

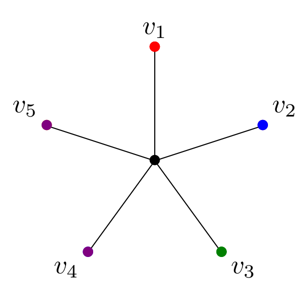


Suppose that in G there is a path from v_1 to v_3, and that the vertices along this path are alternately colored red and green; call such a path a red-green alternating path. Then together with v, this path makes a cycle with v_2 on the inside and v_4 on the outside, or vice versa. This means there cannot be a blue-purple alternating path from v_2 to v_4. Supposing that v_2 is inside the cycle, we change the colors of all vertices inside the cycle colored purple to blue, and all blue vertices are recolored purple. This is still a proper coloring of all vertices of G except v, and now no neighbor of v is blue, so by coloring v blue we obtain a proper coloring of G.

If there is no red-green alternating path from v_1 to v_3, then we recolor vertices as follows: Change the color of v_1 to green. Change all green neighbors of v_1 to red. Continue to change the colors of vertices from red to green or green to red until there are no conflicts, that is, until a new proper coloring is obtained. Because there is no red-green alternating path from v_1 to v_3, the color of v_3 will not change. Now no neighbor of v is colored red, so by coloring v red we obtain a proper coloring of G.

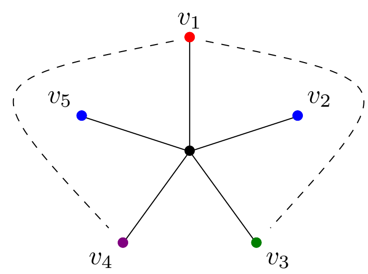
Now suppose $\mathrm{d}(v) = 5$. If the five neighbors of v are colored with three or fewer of the colors, then again v can be colored to give a proper coloring of G with 4 colors.

If the five neighbors are colored with four colors, there are two cases. First, suppose that two vertices with the same color are positioned next to each other, as shown here:



This is very similar to the case $\mathrm{d}(v) = 4$, and left to the reader.

Finally, suppose that the coloring is like this:



If there is no alternating red-green path from v_1 to v_3, we may recolor v_1 green, and proceed as before, changing vertices to red or green as required, and then v may be colored red. If there is such a path, but there is no alternating red-purple path from v_1 to v_4, we may do the same, exchanging red and purple at vertices beginning at v_1.

If this fails, there is both an alternating red-green path from v_1 to v_3 and an alternating red-purple path from v_1 to v_4, shown as dashed curves in the figure. Now starting at v_2, we may swap colors blue and purple, never crossing the red-green path, so that v_4 is still purple; now v_2 is purple. Starting at v_5, we may swap blue and green, never crossing the red-purple path; v_3 is still green and now v_5 is green. Then v may be colored blue.