SAMPLE EXAM 1 ANSWERS

Vectors are denoted by bold letters: $\mathbf{a}$, $\mathbf{v}$, $\mathbf{r}(t)$, etc.

1. Let $\mathbf{a} = \langle 1, 0, -1\rangle$, $\mathbf{b} = \langle -2, 3, 5\rangle$. Find $|\mathbf{a}|$, $\mathbf{a}-\mathbf{b}$, and a unit vector in the same direction as $\mathbf{a}-\mathbf{b}$.

   Answer. $|\mathbf{a}| = \sqrt{2}$; $\mathbf{a}-\mathbf{b} = \langle 3, -3, -6\rangle$; unit vector is $(\mathbf{a}-\mathbf{b})/|\mathbf{a}-\mathbf{b}| = \langle 3/\sqrt{54}, -3/\sqrt{54}, -6/\sqrt{54}\rangle$.

2. Let $\mathbf{v} = \langle 5, -1, 6\rangle$, $\mathbf{w} = \langle -2, 2, -4\rangle$. Find $\mathbf{v}\cdot\mathbf{w}$ and the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$.

   Answer. $\mathbf{v}\cdot\mathbf{w} = -36$; $\cos\theta = \dfrac{-36}{\sqrt{62}\sqrt{24}}$.

3. Which of the pairs of vectors $\{\mathbf{a}, \mathbf{b}\}$, $\{\mathbf{a}, \mathbf{c}\}$, $\{\mathbf{b}, \mathbf{c}\}$ are perpendicular? $\mathbf{a} = \langle 1, 2, 2\rangle$, $\mathbf{b} = \langle 8, -11, 7\rangle$, $\mathbf{c} = \langle -3, 1, 5\rangle$.

   Answer. $\mathbf{a}\perp\mathbf{b}$, $\mathbf{b}\perp\mathbf{c}$.

4. Suppose that $|\mathbf{v}| = 2$, $|\mathbf{w}| = 3$ and the cosine of the angle between $\mathbf{v}$ and $\mathbf{w}$ is $4/5$. Find $|\mathbf{v}\times\mathbf{w}|$.

   Answer. $|\mathbf{v}\times\mathbf{w}| = |\mathbf{v}||\mathbf{w}|\sin\theta = 2\cdot 3\cdot 3/5 = 18/5$. (Use $\sin\theta = \sqrt{1-\cos^2\theta}$.)

5. Find the vector that is called the projection of $\mathbf{a}$ onto $\mathbf{b}$, using $\mathbf{a} = \langle 2, 2, 2\rangle$ and $\mathbf{b} = \langle 1, -1, -1\rangle$.

   Answer. The projection is

   $$\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\frac{\mathbf{b}}{|\mathbf{b}|} = -\frac{2}{3}\langle 1, -1, -1\rangle = \langle -\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\rangle.$$

6. Compute the cross product of $\langle 3, 4, 5\rangle$ and $\langle -5, -1, 2\rangle$.

   Answer. $\langle 13, -31, 17\rangle$.

7. Find an equation of the plane containing the points $(1, 2, 1)$, $(-2, 0, -3)$, $(4, -1, 0)$.

   Answer. $2x + 3y - 3z = 5$.

8. Find an equation for the line that is the intersection of the planes $x + y + z = 2$ and $3x - 2y - z = -5$.

   Answer. $\mathbf{r} = \langle -1/5, 11/5, 0\rangle + t\langle 1, 4-5\rangle$.

9. Find an equation for the plane that is perpendicular to both of the planes $x + y + z = 2$ and $3x - 2y - z = -5$ and contains the point $(1, 1, 1)$.

   Answer. $x + 4y - 5z = 0$.

10. Let $\mathbf{r}(t) = \langle t^2 + 2, t^2 - 4t, 2t\rangle$. Find the tangent line at the point $(6, -4, 4)$.

    Answer. $\mathbf{r} = \langle 6, -4, 4\rangle + t\langle 4, 0, 2\rangle$.

11. Find the curvature of $\mathbf{r}(t)$ from the previous problem as a function of t and also find the curvature at $(6, -4, 4)$.

    Answer.

$$\kappa(t) = \frac{\sqrt{96}}{(4t^2 + (2t-4)^2 + 4)^{(3/2)}}; \quad \kappa(2) = \frac{\sqrt{96}}{(20)^{(3/2)}}.$$

12. Suppose in $\mathbf{r}(t)$ from the previous two problems that t is time. Find the acceleration vector $\mathbf{a}(t)$. Find the scalar accelerations a_T and a_N.

    Answer. $\mathbf{a}(t) = \langle 2, 2, 0\rangle$.

$$a_N = \kappa v^2 = \frac{\sqrt{96}}{(4t^2 + (2t-4)^2 + 4)^{(1/2)}}$$
$$a_T = \frac{\mathbf{r}' \cdot \mathbf{r}''}{|\mathbf{r}'|} = \frac{8t - 8}{(4t^2 + (2t-4)^2 + 4)^{(1/2)}}$$

13. Suppose an object moves so that its velocity vector is $\langle t, t^2, 1\rangle$, and at $t = 0$ it is at the point $(1, 1, 1)$. Where is it at $t = 1$?

    Answer. A vector from the starting point to the ending point is

$$\int_0^1 \mathbf{v}(t)\,dt = \langle t^2/2, t^3/3, t\rangle\Big|_0^1 = \langle 1/2, 1/3, 1\rangle,$$

so a vector to the final point from the origin is $\langle 1, 1, 1\rangle + \langle 1/2, 1/3, 1\rangle = \langle 3/2, 4/3, 2\rangle$.

Alternately, $\mathbf{r}(t) = \langle t^2/2 + 1, t^3/3 + 1, t + 1\rangle$, so $\mathbf{r}(1) = \langle 3/2, 4/3, 2\rangle$.

Thus, at $t = 1$ the object is at $(3/2, 4/3, 2)$.