SAMPLE EXAM 2 ANSWERS

1. Let $f(x,y) = \ln(x^2+y^2)$. Compute the partial derivatives $f_x,\ f_y,\ f_{xx},\ f_{yy},\ f_{xy},\ f_{yx}$.

   Answer.

   $$f_x = 2x/(x^2+y^2) \qquad f_y = 2y/(x^2+y^2) \qquad f_{xy} = f_{yx} = -4xy/(x^2+y^2)^2$$
   $$f_{xx} = (2y^2-2x^2)/(x^2+y^2)^2 \qquad f_{yy} = (2x^2-2y^2)/(x^2+y^2)^2$$

2. Describe the level curves of $f(x,y) = \dfrac{x^2}{9} + \dfrac{y^2}{16}$.

   Answer. All level curves are ellipses with center at the origin. The level curve $k = x^2/9 + y^2/16$ has intercepts at $x = \pm 3\sqrt{k}$ and $y = \pm 4\sqrt{k}$.

3. Find an equation for the tangent plane to $z = e^y \ln x$ at $(1,3,0)$.

   Answer. $f_x = e^y/x$, $f_y = e^y \ln x$. A normal to the tangent plane at $(1,3,0)$ is therefore $\langle e^3, 0, -1\rangle$, and the tangent plane is given by $z = e^3(x-1)$.

4. Use the "two-variable" version of the chain rule to compute $g'(t)$ if $g(t) = f(x,y)$, and $x = t^2$, $y = \cos t$.

   Answer. $g'(t) = f_x(x,y)\cdot 2t + f_y(x,y)(-\sin t)$.

5. Suppose $z = f(x,y)$. At a particular point (x_0,y_0), $\nabla f = \langle a,b\rangle$ is a vector. Describe the significance of both the length and the direction of this vector $\langle a,b\rangle$.

   Answer. The direction of ∇f is the direction in which f increases most rapidly. The rate at which f increases most rapidly is $|\nabla f|$.

6. Find the directional derivative of $f(x,y,z) = x^2yz^3 + xy - z$ at the point $(1,1,1)$ in the direction indicated by the vector $\langle 1,2,3\rangle$.

   Answer. $D_{\mathbf{u}} f = \langle 3,2,2\rangle \cdot \langle 1,2,3\rangle/\sqrt{14} = 13/\sqrt{14}$.

7. Suppose $P(x,y,z) = x^4y - x^2y^3 + y\sin z$ gives the pressure at each point (x,y,z). At the point $(2,1,\pi/4)$, in what direction does the pressure decrease most rapidly? Give your answer as a vector that points in the correct direction.

   Answer. The direction of maximum decrease is given by $-\nabla P = \langle -28, -4 - \sqrt{2}/2, -\sqrt{2}/2\rangle$.

8. The point $(1, -2, 4)$ is on the surface described by $z = x^2y^2 + y \ln x$. Find a vector in one of the two possible directions to go from this point to stay on the level curve $z = 4$.

   Answer. The gradient vector is $\langle 6, -4 \rangle$, so a vector in one of the two desired directions is $\langle 4, 6 \rangle$.

9. Find all critical points for $z = x \sin y$ and classify them as local maximum points, local minimum points, or saddle points.

   Answer. The partial derivatives are

$$f_x = \sin y \qquad f_y = x \cos y \qquad f_{xy} = f_{yx} = \cos y$$
$$f_{xx} = 0 \qquad f_{yy} = -x \sin y$$

   The first partial derivatives are both zero when $x = 0$ and $y = n\pi$, for any integer n. $D(0, n\pi) = -\cos^2(n\pi) = -1 < 0$, so all are saddle points.

10. Find the maximum and minimum values of $f(x, y) = x^2 - y^2$ above the curve given by $x^2 + 2y^2 = 1$.

    Answer. From the second equation, $x^2 = 1 - 2y^2$, so we want to find the maximum and minimum values of $g(y) = 1 - 3y^2$ when $-1/\sqrt{2} \leq y \leq 1/\sqrt{2}$. $g'(y) = -6y$ so there is a critical point when $y = 0$. The values of g to check are $g(0) = 1$ and $g(\pm 1/\sqrt{2}) = -1/2$. The maximum of f is 1 and the minimum is $-1/2$.