SAMPLE EXAM 3 SOLUTIONS

1. Compute: $\displaystyle\iint_R xe^{xy}\,dA$, $R = [0, 1] \times [0, 1]$.

$$\int_0^1 \int_0^1 xe^{xy}\,dy\,dx = \int_0^1 e^{xy}\Big|_0^1 dx = \int_0^1 e^x - 1\,dx = e^x - x\Big|_0^1 = e - 1 - 1 = e - 2$$

2. Compute: $\displaystyle\iint_R 1/x\,dA$, $R = \{(x, y) \mid 1 \le y \le e, y^2 \le x \le y^4\}$.

$$\int_1^e \int_{y^2}^{y^4} \frac{1}{x}\,dx\,dy = \int_1^e \ln y^4 - \ln y^2 dy = \int_1^e 4\ln y - 2\ln y dy = \int_1^e 2\ln y\,dy$$
$$= 2\big[y\ln y - y\big]_1^e = 2(e - e - 0 + 1) = 2$$

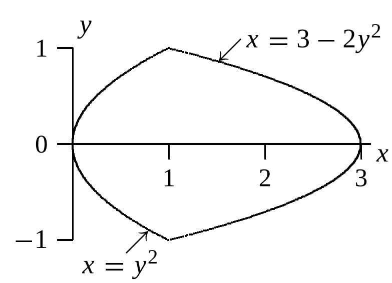
3. Compute: $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy$.

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy = \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\,dy\,dx = \int_0^1 x^2\sqrt{x^3+1}\,dx$$
$$= \frac{1}{3}\frac{2}{3}(x^3+1)^{3/2}\Big|_0^1 = \frac{2}{9}2^{3/2} - \frac{2}{9}$$

4. Find the volume under $z = x^2 + y^2$ and above $[-2, 2] \times [-3, 3]$.

$$\int_{-2}^2 \int_{-3}^3 x^2+y^2\,dy\,dx = \int_{-2}^2 x^2y+\frac{y^3}{3}\Big|_{-3}^3 dx = \int_{-2}^2 6x^2+18\,dx = \frac{6x^3}{3}+18x\Big|_{-2}^2 = 104$$

5. Find the volume under $z = y^2 - x + 3$ and above the region shown.



$$\int_{-1}^1 \int_{y^2}^{3-2y^2} y^2 - x + 3\,dx\,dy =$$

$$\int_{-1}^1 y^2x - \frac{x^2}{2} + 3x\Big|_{y^2}^{3-2y^2} dy = \frac{9}{2}\int_{-1}^1 -y^4 + 1\,dy = \frac{36}{5}$$

6. Find the volume under $z = xy$ and above the region inside $r = 1 + \cos\theta$ in the first quadrant.

$$\int_0^{\pi/2}\int_0^{1+\cos\theta} \sin\theta\cos\theta r^3\,dr\,d\theta = \int_0^{\pi/2} \frac{(1+\cos\theta)^4}{4}\sin\theta\cos\theta\,d\theta$$

Now use $u = 1 + \cos\theta$:

$$\int_1^2 \frac{u^4}{4}(u-1)\,du = \frac{1}{4}\left(\frac{2^6-1}{6} - \frac{2^5-1}{5}\right) = \frac{43}{40}$$

7. A flat plate has the shape bounded by the parabola $y = 9 - x^2$ and the x-axis; the density is given by $\rho(x, y) = x^2 y$. Find the total mass and the y coordinate of the center of mass.

$$M = \int_{-3}^{3}\int_0^{9-x^2} x^2 y\,dy\,dx = \frac{1}{2}\int_{-3}^{3} x^2(9-x^2)^2\,dx = \frac{1}{2}\int_{-3}^{3} 81x^2 - 18x^4 + x^6\,dx =$$

$$3^6 - \frac{18\cdot 3^5}{5} + \frac{3^7}{7} = \frac{5832}{35}$$

$$\bar{y} = \frac{1}{M}\int_{-3}^{3}\int_0^{9-x^2} x^2y^2\,dy\,dx = \frac{1}{3M}\int_{-3}^{3} x^2(9-x^2)^3\,dx =$$

$$\frac{1}{3M}\int_{-3}^{3} 3^6x^2 - 3^5x^4 + 3^3x^6 - x^8\,dx = \frac{2}{3M}\left(3^8 - \frac{3^{10}}{5} + \frac{3^{10}}{7} - 3^7\right) = \frac{23328}{35M} = 4$$

8. Compute: $\displaystyle\int_0^2\int_0^{\sqrt{9-x^2}}\int_0^{x^2} yz\,dy\,dz\,dx.$

$$\int_0^2\int_0^{\sqrt{9-x^2}}\int_0^{x^2} yz\,dy\,dz\,dx = \frac{1}{2}\int_0^2\int_0^{\sqrt{9-x^2}} x^4 z\,dz\,dx = \frac{1}{4}\int_0^2 x^4(9-x^2)\,dx =$$

$$\frac{9\cdot 2^3}{5} - \frac{2^5}{7} = \frac{344}{35}$$

9. Compute: $\iiint\limits_R x^3 + xy^2\,dV$, where R is the three dimensional region in the first octant that is under $z = 1 - x^2 - y^2$.

$$\int_0^{\pi/2}\int_0^1\int_0^{1-r^2} \cos\theta\, r^4\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 \cos\theta\, r^4(1-r^2)\,dr\,d\theta =$$
$$\int_0^{\pi/2} \cos\theta \left[\frac{r^5}{5} - \frac{r^7}{7}\right]_0^1 d\theta = \frac{2}{35}\sin\theta\Big|_0^{\pi/2} = \frac{2}{35}$$

10. Find the mass of a hemisphere of radius 1 if the density is $\rho(x, y, z) = z$, assuming that the sphere is centered at the origin and the hemisphere is the upper half.

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho\cos\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{4}\int_0^{2\pi}\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\,d\theta =$$
$$\frac{1}{4}\int_0^{2\pi}\frac{1}{2}\left[\sin^2\phi\right]_0^{\pi/2} d\theta = \frac{1}{8}\int_0^{2\pi} d\theta = \frac{\pi}{4}$$