# SAMPLE EXAM 4 SOLUTIONS

1. Compute $\int_C xy\,ds$, where $C$ is given by $\langle \sin\theta, \cos\theta\rangle$, $0 \le \theta \le \pi/2$.

   **Answer.** $\mathbf{r}' = \langle \cos\theta, -\sin\theta\rangle$, $|\mathbf{r}'| = \mathbf{1}$, so $\int_C xy\,ds = \int_0^{\pi/2} \sin\theta\cos\theta\,d\theta = 1/2$.

2. Explain how you can tell that $\mathbf{F} = \langle 3x^2\cos y, -x^3\sin y\rangle$ is conservative. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $C$ is $\langle \cos t, t^2\rangle$, $0 \le t \le 1$.

   **Answer.** $\mathbf{F}$ is conservative because $\mathbf{F} = \nabla(x^3\cos y)$.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = x^3\cos y\Big|_{(1,0)}^{(\cos(1),1)} = \cos^4(1) - 1.$$

3. Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F} = \langle x^2y^2, 3x + xy\rangle$ and $C$ is the square $(0,0) \to (1,0) \to (1,1) \to (0,1) \to (0,0)$.

   **Answer.** Since the curve $C$ is closed, we may use Green's Theorem:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1\int_0^1 3 + y - 2x^2y\,dy\,dx = 3 + \frac{1}{2} - \frac{1}{3} = \frac{19}{6}.$$

4. Compute $\nabla\times\mathbf{F}$, $\mathbf{F} = \langle x\cos z, y\cos z, \sin z\rangle$. Is $\mathbf{F}$ conservative?

   **Answer.** $\nabla\times\mathbf{F} = \langle y\sin z, -x\sin z, 0\rangle$. $\mathbf{F}$ is not conservative because $\nabla\times\mathbf{F}$ is not the zero vector.

5. Compute $\nabla\cdot\mathbf{F}$, $\mathbf{F} = \langle x\cos z, y\cos z, \sin z\rangle$.

   **Answer.** $\nabla\cdot\mathbf{F} = \cos z + \cos z + \cos z = 3\cos z$.

6. Set up a double integral for the surface area of $\mathbf{r} = \langle u^2, u^2 - v, v^3\rangle$, $0 \le u \le 1$, $0 \le v \le 1$.

   **Answer.** $\mathbf{r}_u = \langle 2u, 2u, 0\rangle$, $\mathbf{r}_v = \langle 0, -1, 3v^2\rangle$, $\mathbf{r}_u\times\mathbf{r}_v = \langle 6uv^2, -6uv^2, -2u\rangle$, and $|\mathbf{r}_u\times\mathbf{r}_v| = \left(36u^2v^4 + 36u^2v^4 + 4u^2\right)^{1/2}$. The surface area is then

$$\int_0^1\int_0^1 \left(36u^2v^4 + 36u^2v^4 + 4u^2\right)^{1/2}\,du\,dv.$$

7. Compute $\iint_S \mathbf{F} \cdot d\,\mathbf{S}$, where $\mathbf{F} = \langle y, z, x\rangle$ and $S$ is the surface $z = x^2 + y^2$ above the interior of the square with corners $(0, 0)$, $(1, 0)$, $(1, 1)$, $(0, 1)$.

**Answer.** Use $\mathbf{r} = \langle x, y, x^2 + y^2\rangle$, so $\mathbf{r}_x = \langle 1, 0, 2x\rangle$, $\mathbf{r}_y = \langle 0, 1, 2y\rangle$, and $\mathbf{r}_x \times \mathbf{r}_y = \langle -2x, -2y, 1\rangle$. Then

$$\iint_S \mathbf{F} \cdot d\,\mathbf{S} = \int_0^1 \int_0^1 \langle y, x^2 + y^2, x\rangle \cdot \langle -2x, -2y, 1\rangle\, dy\, dx$$

$$= \int_0^1 \int_0^1 -2xy - 2x^2y - 2y^3 + x\, dy\, dx = -5/6.$$

8. Compute $\iint_S \nabla \times \mathbf{F} \cdot d\,\mathbf{S}$, where $\mathbf{F} = \langle z, y, x\rangle$ and $S$ is the surface $z = x^2 + y^2$ above the disk $x^2 + y^2 \leq 1$.

**Answer.** We may use Stokes' Theorem; the boundary is given by $\mathbf{r} = \langle \cos\theta, \sin\theta, 1\rangle$, $0 \leq \theta \leq 2\pi$, so $\mathbf{r}' = \langle -\sin\theta, \cos\theta, 0\rangle$. Then

$$\iint_S \nabla \times \mathbf{F} \cdot d\,\mathbf{S} = \int_0^{2\pi} \langle 1, \sin\theta, \cos\theta\rangle \cdot \langle -\sin\theta, \cos\theta, 0\rangle\, d\theta$$

$$= \int_0^{2\pi} -\sin\theta + \sin\theta\cos\theta\, d\theta = 0.$$

9. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = \langle z^2, y, x\rangle$ and $C$ is the triangle $(1, 0, 0) \to (0, 1, 0) \to (0, 0, 1) \to (1, 0, 0)$.

**Answer.** Since the curve $C$ is closed we may use Stokes' Theorem. The curve is the boundary of the surface given by $\mathbf{r} = \langle x, y, 1 - x - y\rangle$, $0 \leq x \leq 1$, $0 \leq y \leq 1 - x$. $\mathbf{r}_x \times \mathbf{r}_y = \langle 1, 1, 1\rangle$, so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\,\mathbf{S} = \int_0^1 \int_0^{1-x} 2(1 - x - y) - 1\, dy\, dx = -1/6.$$

10. Compute $\iint_S \mathbf{F} \cdot d\,\mathbf{S}$, where $\mathbf{F} = \langle x^2z, z^2y, y^2x\rangle$ and $S$ is the surface of the cube with corners $(0, 0, 0)$, $(1, 0, 0)$, $(1, 0, 1)$, $(0, 0, 1)$, $(0, 1, 0)$, $(1, 1, 0)$, $(1, 1, 1)$, $(0, 1, 1)$, oriented outward.

**Answer.** The surface is closed so we use the divergence theorem.

$$\iint_S \mathbf{F} \cdot d\,\mathbf{S} = \int_0^1 \int_0^1 \int_0^1 2xz + z^2\, dy\, dx\, dz = 5/6.$$