Introduction to Chaotic Dynamical Systems

Prof Hundley

Spring 2024

Introduction

The Iteration Game

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- Take its square root.

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- $0, 1$ are called **fixed points**.
- All other numbers got closer and closer and closer to the value 1.

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Using function iteration,

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$$L^2 - L = 0 \quad \Rightarrow \quad L(L-1) = 0$$

so $L = 0$ or $L = 1$.

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If $x_0 > 0$, then $x_n \to \sqrt{5}$.

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- A rule for how to move them

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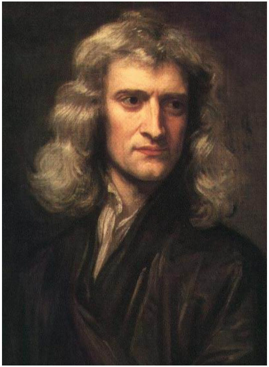
A **differential equation** is a continuous dynamical system.

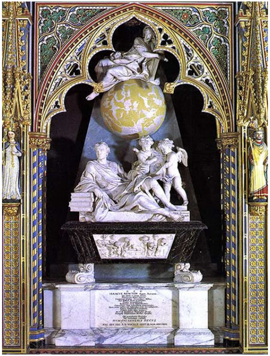
Goal

Is it possible to predict the behavior of all points for all time (forward and back), given the current state and the rules?

A Few Mathematicians



Sir Isaac Newton
1643-1727
Developed Calculus
Motion of the Planets



Grave at Westminster Abbey

Newton attended Trinity College at Cambridge, in particular in 1665 when the Great Plague started in London. He fled home, and during those years came up with the start of calculus.
(Newton, with or without the plague, was one of the greatest mathematicians of all time)

Pierre-Simon Laplace

- Best known for: investigating the stability of the solar system
- Laplace's equation: $u_{xx} + u_{yy} = 0$
- The Laplace transform in differential equations.
- Determinism
- Probability

Laplace's Demon

"We may regard the present state of the universe as the effect of its past and the cause of its future. An intellect which at any given moment knew all of the forces that animate nature and the mutual positions of the beings that compose it, if this intellect were vast enough to submit the data to analysis, could condense into a single formula the movement of the greatest bodies of the universe and that of the lightest atom; for such an intellect nothing could be uncertain and the future just like the past would be present before its eyes"

– marquis de Laplace (1814)

Henri Poincaré

- b. 1854-d. 1912
- King Oscar II of Sweden
  Contest to solve $N-$body problem (1890)
- Prize to Poincaré
  His paper led to what is now known as Chaos- A crossing of stable and unstable manifolds
- Poincaré Conjecture (1904)
  Every simply connected closed 3-manifold is homeomorphic to the 3-sphere.
  Now solved (2006)

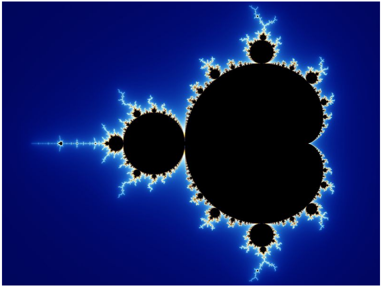
Pierre Fatou: 1878-1929

Gaston Julia: 1893-1978

Independently worked on iterations of complex map, 1920s

$$z \to z^2 + c$$

Benoit Mandelbrot
1924 - 2010

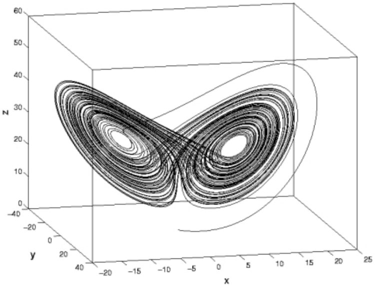


A catalog of Julia Sets, which was basically forgotten until the 1970s.

George David Birkhoff
1884-1944

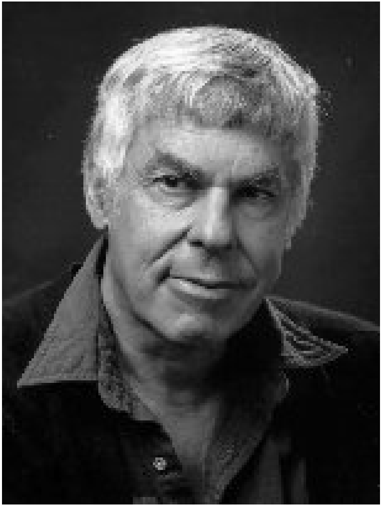
- Continued and Proved some of Poincaré's work
- Statistical physics
- Ergodic Theorem

Edward N. Lorenz
1917-2008



The Lorenz Attractor

Meteorology; Main paper: "Deterministic nonperiodic flow" (1963)
Coined "Butterfly Effect" for Sensitive Dependence

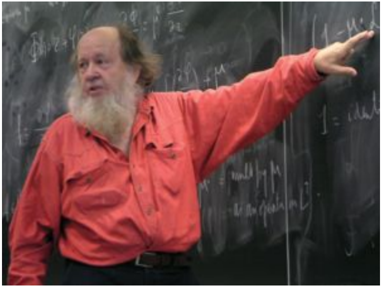


Steve Smale
1930 (Age 93)

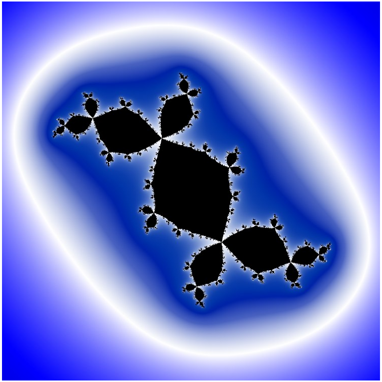
- Proved Poincaré Conjecture for dimensions greater than 5.
- Developed symbolic dynamics while down in Rio
- Fields Medal in 1966

Sir Robert May
1936-2020
Ecology; Logistic Map

Mitchell Feigenbaum
Physics
Feigenbaum constant



Adrien Douady
1935-2006
Worked with his student
John Hubbard at Cornell



The Douady Rabbit
A Julia Set

Robert Devaney (Age 75)
Boston University
Student of Smale
Exponential Maps

James Yorke (Age 82)
University of Maryland
"Period 3 Implies Chaos"