Sample Exam 1 Solutions

This sample exam is not intended to reflect the length of the actual exam. Rather, it is intended to show the types of questions you should expect.

1. True or False (and a short reason):

   (a) The function $f(x) = x^2 + 1$ has an attracting fixed point at the origin.
   SOLUTION: The origin is not a fixed point, since $f(0) \neq 0$, so FALSE.

   (b) Let $(001001\ldots)$ and $(1111\ldots)$ be two sequences in the sequence space Σ. Then the distance between these points is $2/3$.
   SOLUTION: The distance is

   $$\frac{1}{2^0} + \frac{1}{2^1} + \frac{0}{2^2} + \frac{1}{2^3} + \cdots = 1 + \frac{1}{2} + \frac{1}{8} + \cdots > 1 > \frac{2}{3}$$

   so FALSE.

   (c) The point with ternary expansion $0.010101\ldots$ is an endpoint of the Cantor Middle-Thirds set.
   SOLUTION: We said that all the points in Cantor set can be built with 0's and 2's, but then can be expressed in strings of 0's and 1's. It's not clear from the question which this is- but in either case, the answer would be FALSE:

   - If we think of this point as coming from $0.02020202\cdots$, then any point that's an endpoint will end in all zeros or all 2's.
   - If we think of this point base 3, then it is not in the Cantor set, since it is not expressed in terms of 0's and 2's.
   - Also in the second case, we can determine this point to be $1/8$, which is between $1/9$ and $2/9$ (and so would be removed).

   (d) The point 1 lies on a cycle of period 2 for the function $f(x) = x^2 - 1$.
   SOLUTION: $f(1) = 0$ and $f(0) = -1$ and $f(-1) = 0$, so the point $x = 1$ is **eventually** periodic with period 2 (so overall, FALSE).

2. Quickies (Answers only- no partial credit)

   (a) The function $f(x) = |x|$ has eventually fixed points at:
   SOLUTION: all $x < 0$.

   (b) For which value(s) of k does the function $f(x) = k\arctan(x)$ have an attracting fixed point at the origin?
   SOLUTION:

   $$f'(x) = \frac{k}{1 + x^2} \quad \Rightarrow \quad |f'(0)| < 1 \text{ for } |k| < 1$$

(c) List all cycles of prime period 4 for the shift map σ on Σ.
SOLUTION: The repeating sequence ($\overline{1001}$) is:

$$\overline{1001} \quad \to \quad \overline{0011} \quad \to \quad \overline{1100}$$

The other two are given by the orbits of ($\overline{0001}$) and ($\overline{0111}$).

(d) The number whose ternary expansion is $0.12121212\ldots$ is:
SOLUTION:

$$\begin{array}{rl} -S & = 0.121212\ldots \\ 9S & = 12.121212\ldots \\ \hline 8S & = (12)_3 \end{array} \quad \Rightarrow \quad 8S = 1 \cdot 3^1 + 2 \cdot 3^0 = 5 \quad \Rightarrow \quad S = 5/8$$

(e) Given an example of a cycle of period 2 for the doubling map $D(x)$ ($D(x)$ would be defined on the exam).
SOLUTION: $1/3, 2/3$ is a cycle of period 2 for the doubling map.

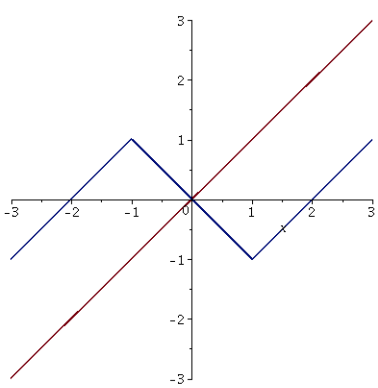
3. Use graphical analysis to give a complete orbit analysis of the piecewise defined function:

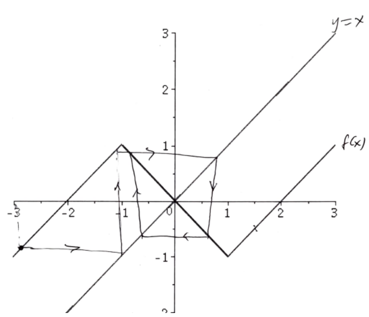
$$f(x) = \begin{cases} x+2 & \text{if } x \le -1 \\ -x & \text{if } -1 < x < 1 \\ x-2 & \text{if } x \ge 1 \end{cases}$$

List all fixed points and cycles and tell if they are attracting, repelling, or neutral. . List all points whose orbits tend to cycles or fixed points, all points whose orbits tend to infinity, and all points that are eventually fixed or periodic.

SOLUTION: We should think of two regions- One region is inside the box $-1 < x < 1, -1 < y < 1$. Once the orbit goes into this box, then $f(x) = -x$ makes every point (except $x = 0$) into a 2-cycle. Outside the box, the orbit takes the point towards the box.

To the left is the cobweb diagram, and to the right is a sample orbit.





As a summary, the only fixed point is at the origin, and it is a neutral fixed point. All the points in the "box" are period 2 points. The points $x = \pm 2$ get mapped to zero, so they are eventually fixed. Think now about the points that get mapped to ± 2. This would be $x = 4$ (the orbit would be $4, f(4) = 2,\ f(2) = 0,\ 0, \ldots$. In fact, if $x = 2, 4, 6, 8, \ldots$, then x is eventually fixed. All other points are eventually periodic.

4. Give the definition:

   (a) Homeomorphism: A function f is a homeomorphism if f is 1-1, onto, continuous, and its inverse is also continuous (or, f is a bijection, continuous and its inverse is continuous).

   (b) Conjugacy: Functions F and G are conjugate if there is a homeomorphism h such that
   $$h \circ F = G \circ h$$

   (c) Saddle node bifurcation: A dynamical system (X, f) with parameter λ undergoes a saddle node bifurcation at $\lambda = \lambda_0$, if, on one side of λ_0 there are no fixed points, then on the other side of λ_0, there are two. To be more precise, see the definition at the beginning of 6.2, and the figure on the next page.

   (d) One to one: If $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

   (e) Dense: For example, when is set A dense in set B?

   Set A is dense in B if every point in B is either in A or is a limit point of A. Alternatively, for every point in $b \in B$, and there is a point of A in every $\epsilon-$neighborhood of b.

5. Show that σ is continuous on Σ.

   SOLUTION: Main idea is that, to guarantee that $\sigma(\mathbf{s})$ is within $1/2^n$ to $\sigma(\mathbf{t})$, then the symbol strings $\mathbf{s}, \mathbf{t}$ must share at least $n + 2$ places.

   That is, let $\epsilon > 0$ be given, and let n be an integer so that $1/2^n < \epsilon$. Now, set $\delta = 1/2^{n+1}$. If $\mathbf{s}, \mathbf{t}$ are within $1/2^{n+1}$, they share $n + 2$ places. Applying the shift map to $\mathbf{s}, \mathbf{t}$ means that they now share $n + 1$ places, so that the distance between them is less than $1/2^n$.

6. Examples. Give an example of a continuous function F which has the following property (or prove that such a function does not exist). Note: a different function for each property.

   (a) A repelling fixed point at the origin and an attracting fixed point at 1 (and no other fixed points or cycles).

   SOLUTION: We can try a piecewise linear function, or use $f(x) = \sqrt{x}$ as two possible examples.

   (b) All orbits escaping to infinity except for a repelling cycle of period two at 0 and at 1.

   SOLUTION: If we have a two cycle at $(0, 0)$ and $(1, 1)$, then what are the requirements of our function?

- $f(0) = 1$ and $f(1) = 0$
  (when cobwebbing, this will form a box $(0,0), (0,1), (1,1), (1,0)$).
- Repelling: $|f'(0)| > 1$ and $|f'(1)| > 1$.

But really, the main question will be: Can your function go through $(1,0)$ and $(0,1)$ without intersecting the line $y = x$? If the function is continuous (and we typically assume that it is), the answer is "No", and that intersection will be a fixed point (therefore, there is no way we can have all other orbits escape to infinity except for the two-cycle).

7. State the Mean Value Theorem. Use it to discuss why, if $F(p) = p$ and $|F'(p)| < 1$, then there is an interval about p in which all orbits tend to p.

SOLUTION: The Mean Value Theorem says that, if f is continuous on $[a,b]$ and differentiable on (a,b), then there is a c in $[a,b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Or, more to the point for us, $|f(b) - f(a)| = |f'(c)||b - a|$.

Therefore, if $f(p) = p$ and $|f'(p)| < 1$ for all points in an interval about p, then we assume there is a number $\mu < 1$ so that $f'(p) < \mu < 1$. Therefore,

$$|f(x) - f(p)| = |f'(c)||x - p| \le \mu|x - p|$$

Or,

$$|f(x) - p| \le \mu|x - p| \quad \Rightarrow \quad |f^n(x) - p| < \mu^n|x - p|$$

so that the orbit converges to p.

8. Consider the family of functions $f_k(x) = k(e^x - 1)$ where $k > 0$. A bifurcation occurs for this family near 0 when $k = 1$.

   (a) For which values of k is the fixed point at the origin attracting? For which values is it repelling?

   SOLUTION: Take the derivative at the origin: $f'(x) = k\mathrm{e}^x$, so $f'(0) = k$. Therefore, the origin is attracting if $k < 1$ (note that we're told that $k > 0$, so we don't need absolute value), repelling if $k > 1$.

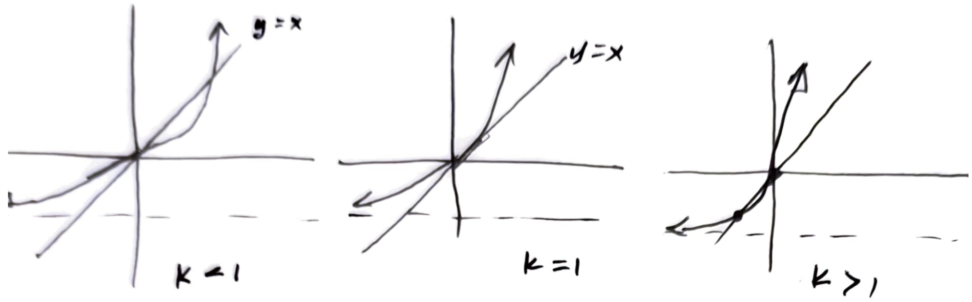
   (b) Describe the bifurcation that occurs at $k = 1$.

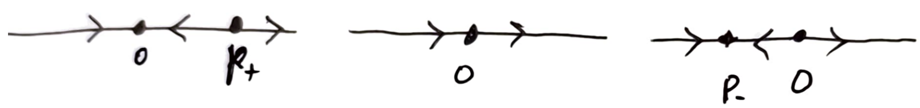
   (Note: for the next couple of problems, I would give you the relevant graphs if we were doing this in class. As a take home problem or review problem, you can use the computer). Using a graph, we see that if $k > 1$, there are two fixed points, one at $x = 0$ and one that is $x < 0$. When $k = 1$, we have only the one fixed point at $x = 0$, and if $k > 1$, then we have two fixed points- one at $x = 0$, and $x > 0$.

   We note that, for $k > 1$, the origin is repelling and the negative fixed point is attracting. At $k = 1$, we have only the neutral fixed point, and for $k < 1$, the origin becomes attracting and the fixed point $x > 0$ is repelling. (See the graphs).

(c) Use graphical analysis on these functions at, before, and after the bifurcation. (graph below)

(d) Sketch the phase portraits for these functions at, before, and after the bifurcation. (graph below)





9. Discuss the three parts to Devaney's definition of a chaotic dynamical system, and use Σ as the primary example.

SOLUTION: The three parts to chaos (using Devaney's definition) are that (i) periodic points are dense, (ii) The dynamical system is transitive, and (iii) the system has sensitive dependence on intial conditions. Later, it was determined that points (i) and (ii) will imply the third, so we focus on the first two for Σ.

(a) Given any string in Σ, there is a convergent string of periodic orbits that converge to it. That is, given $s_0 s_1 s_2 \ldots$, then the sequence:

$$(\overline{s_0}), (\overline{s_0 s_1}), (\overline{s_0 s_1 s_2}), \ldots$$

converges to the string.

(b) There is the orbit of a single point that is dense in Σ, so while that is not the definition of transitive, it will imply that the system is transitive. The string was constructed of all strings of length 1, followed by all strings of length 2, followed by all strings of length 3, and so on.

10. Intuitively, why was the itinerary map $S(x)$ an "onto" map? (You don't need to give a formal proof, I'm looking for a verbal description- think about how we showed it).

SOLUTION: We recall that "onto" in this case means that every string in Σ comes from the itinerary of (at least) one point in Λ. To show this, we constructed the sets

I_{s_0}, $I_{s_0 s_1}$, $I_{s_0 s_1 s_2}$, and so on, and we showed that these sets were nested and closed-therefore, the point whose orbit corresponded to our string is the point that is in the infinite intersection of these sets (we have a theorem that says the infinite intersection of nested sets is never empty). We also showed that the intersection goes to a single point, but that wasn't necessary for "onto".