Homework Compilation
Chapter 11

This includes the homework previously assigned since Oct 16th, plus some extra material.

1. (Lyapunov Exponents) If p is a fixed point for f, what is the Lyapunov exponent for any point whose orbit is attracted to p?

   SOLUTION: We recall the definition of the Lyapunov Exponent for the orbit of x_1:

   $$h(x_1) = \lim_{n\to\infty} \frac{\ln|f'(x_1) + \cdots + \ln|f'(x_n)|}{n}$$

   where $f(x_1) = x_2$, $f(x_3) = x_3$, etc. We will also use the Lemma that we discussed in class: If $\{s_i\}_{i=1}^{\infty}$ is a sequence that converges to s, then

   $$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} s_i = s$$

   Back to the homework question: If x_1 is any point that converges to the attracting fixed point p, then we will assume that $f'(x)$ is continuous so that, by continuity,

   $$x_n \to p \quad \Rightarrow \quad f'(x_n) \to f'(p) \quad \Rightarrow \quad \ln|f'(x_n)| \to \ln|f'(p)|$$

   We will also need to assume that $f'(x_n) \neq 0$ and $f'(p) \neq 0$. Given that, we can use the Lemma directly:

   $$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \ln|f'(x_i)| = \ln|f'(p)|$$

2. (Sect 11, Exercise 1) Can a continuous function on $\mathbb{R}$ have a periodic point of period 48 and not one of period 56? Why?

   SOLUTION: Do a prime factorization:

   $$48 = 2^4 \cdot 3 \qquad 56 = 2^3 \cdot 7$$

   In the Sarkovskii ordering, the odd numbers are in the first row, two times the odds are second, 2^2 times the odds are third, so 56 is in the fourth row, and 48 is in the fifth:

   $$\begin{array}{cccc} 3 & 5 & 7 & \ldots \\ 2\cdot 3 & 2\cdot 5 & 2\cdot 7 & \ldots \\ 2^2\cdot 3 & 2^2\cdot 5 & 2^2\cdot 7 & \ldots \\ 2^3\cdot 3 & 2^3\cdot 5 & \mathbf{2^3\cdot 7} & \ldots \\ \mathbf{2^4\cdot 3} & 2^4\cdot 5 & 2^4\cdot 7 & \ldots \end{array}$$

   Therefore, a (continuous) function could have a periodic point with period 48 but not 56 (however, if it had a period 56 point, it would have to have a period 48 point).

3. (Sect 11, Exercise 3) Give an example of a function $F : [0,1] \to [0,1]$ that has a periodic point of period 3 and NO other periods (Hint: Can this happen?).

   The main thing is that the function cannot be continuous. For example,

$$F(x) = \begin{cases} 1/3 & \text{if } x = 0 \\ 2/3 & \text{if } x = 1/3 \\ 0 & \text{elsewhere} \end{cases}$$

   Notice that F does not intersect $y = x$, and the orbit of every point is eventually periodic (so in particular, the orbit could not be of any other period).

4. (Sect 11, Exercise 4) Let f, g be piecewise linear functions where each has a cycle of period 4 given by $\{0, 1, 2, 3\}$. In particular, define f as:

$$f(0) = 1 \quad f(1) = 2 \quad f(2) = 3 \quad f(3) = 0$$

   and g:

$$g(0) = 3 \quad g(1) = 2 \quad g(2) = 0 \quad g(3) = 1$$

   One of these functions has cycles of all other periods, and one has only periods $1, 2, 4$. Which is which? (They are easy to plot, but the graphs are given in Fig 11.16)

   SOLUTION:

   For the function f, we see that:

$$[0,1] \to [1,2] \to [2,3] \to [0,3]$$

   Therefore, $f^3([0,1]) \supset [0,3]$, and so there must be a period 3 point in the interval $[0,1]$. Is it prime period 3? Yes- It cannot be a fixed point since $f([0,1]) = [1,2]$ and 1 is on the 4-cycle. Similarly, it is not period 2, so the period three point is a prime period 3.

   On the other hand, for the function g,

$$[0,1] \to [2,3] \to [0,1]$$

   So there is a period 2 point in $[0,1]$. There is a 4-cycle (given), and the fixed point is in the interval $[1,2]$. So this function will be the second case (there will be no other cycles).

5. (Added from the board) If f is piecewise linear so that $f(1) = 2$, $f(2) = 3$ and $f(3) = 1$, find the interval (explicitly) that contains a period 4 point (The idea is to run through the proof of the Period 3 theorem to find intervals I_0, I_1, A_1, A_2, A_3, and A_4).

   First, $I_0 = [1,2]$ and $I_1 = [2,3]$. Now, A_1 was the preimage of I_1 in I_1 (graphically, locate I_1 on the $y-$axis and locate all the x's in I_1 that map to it). The preimage of $y = 2$ is:

$$x + 1 = 2 \text{ or } -2x + 7 = 2 \quad \Rightarrow \quad x = 1 \text{ or } x = \frac{5}{2}$$

Graphically, we see that the preimage of 3 is 2. Therefore,

$$A_1 = f^{-1}([2,3]) \quad \Rightarrow \quad A_1 = \left[2, \frac{5}{2}\right]$$

Now, A_2 is the preimage of A_1 in the interval I_1, so locate A_1 on the $y-$axis to help us visualize what we need. Algebraically, we already have the preimage of $x = 2$ (which was either 1 or 5/2), so now we just need the preimage of $x = 5/2$ using the second line:

$$-2x + 7 = \frac{5}{2} \quad \Rightarrow \quad x = \frac{9}{4} \quad \Rightarrow \quad A_2 = \left[\frac{9}{4}, \frac{5}{2}\right]$$

The interval A_3 is the preimage of A_2 in I_0, and A_4 is the preimage of A_3 in I_1:

$$A_3 = \left[\frac{5}{4}, \frac{3}{2}\right] \qquad A_4 = \left[\frac{11}{4}, \frac{23}{8}\right] = [2.75, 2.875]$$

In conclusion, we have constructed an interval in which the period 4 point resides, since

$$f^4([2.75, 2.875]) = [2,3] \qquad \text{or} \qquad f^4(A_4) = I_1$$

6. (Sect 11, Exercise 7) If f is piecewise linear so that

$$f(0) = 4 \quad f(1) = 5 \quad f(2) = 3 \quad f(3) = 0 \quad f(4) = 1 \quad f(5) = 2$$

(See Fig 11.17(a)), prove that there is a cycle of period 6 but no cycles of any odd period.

NOTE: First look to see if you can find why- For example, look for a three or five cycle. The difficult part is in the interval $[2,3]$, since f has a fixed point there (it is sufficient if you can see that there is no three or five cycle).

What you should see is something like:

$$[0,1] \to [4,5] \to [1,2] \to [3,5] \to [0,2] \to [3,5] \to [0,2]$$

Notice what is happening? If we continue, we would see the same pattern...

Since $f^3([0,2]) = [3,5]$ and $f^3([3,5]) = [0,2]$, there are no three cycles (or cycles of any odd period) in $[0,2]$ or in $[3,5]$. The only interval left is $[2,3]$. This interval contains the fixed point.

7. What was so special about *Period 3*? Why not period 4? Period 7?

   Period three enabled us to separate the intervals between the points in the cycle in a special way- Into I_0 and I_1 so that

   $$Q(I_0) = I_1 \qquad Q(I_1) = I_0 \cup I_1$$

   Once we are here, we can build up a cycle of any period.

8. Let F map points on the circle to the circle by rotating them $\pi/3$ radians.

   Error: The rotation should be $2\pi/3$

   This is the note at the top of p. 139

   (a) Are there any period 3 points? All points are period 3

   (b) Are there any other (prime) periodic points? There are no other periodic points

   (c) Does this contradict the Period Three theorem? No- F must map the real line to the real line and be continuous.

9. Show that, if f is differentiable and decreasing for all x, then f^3 is decreasing.

   Differentiate f^3 and show that the derivative is negative:

   $$\frac{d}{dt} f(f(f(t))) = f'(f(f(t)))f'(f(t))f'(t)$$

   Since f is decreasing, f' is negative, and the product of the three negative numbers in negative.

10. Show that, if f is strictly decreasing and there is one fixed point of f, then there can be no other fixed points.

    NOTE: It is OK to assume that f is differentiable:

    Suppose there were two fixed points, a and b. Then by the MVT,

    $$\frac{f(b) - f(a)}{b - a} = f'(c)$$

    for some c between a and b. Simplifying, this means that $f'(c) = 1$- But $f' < 0$ at all values of x. Therefore, we cannot have two fixed points.

11. Define Σ_a as the subset of Σ_3 (the space of all symbol strings containing $0, 1$ or 2) where there are no double integers in the symbol string- That is, Σ_a cannot have 00, 11 or 22.

(a) Show that Σ_a is closed in Σ (Prove that the complement is open- Look at how we did it in class)

Let point $s \in (\Sigma_a)^c$. Then s must have a pair of zeros, a pair of ones and/or a pair of twos. Let $s_{k-1} = s_k$ be the first positions of the pair. Let $\epsilon > 0$ be some arbitrary small number- We will show that every point that is within ϵ of s must also be in the complement.

First, we need to modify the metric slightly (we did this in class):

$$d(s,t) = \sum_{n=0}^{\infty} \frac{|s_n - t_n|}{3^n}$$

In general, if we have Σ_m (the space of strings using symbols $0, 1, 2, \ldots, m-1$) then the metric is typically taken to be:

$$d(s,t) = \sum_{n=0}^{\infty} \frac{|s_n - t_n|}{m^n}$$

This ensures that the sum converges (there are many other possible metric, however).

Back to the problem: Let k be chosen so that $\frac{1}{3^k} < \epsilon$. Then by the Proximity theorem (slightly modified), $d(s,t) < 1/3^k$ if and only if the first k symbols of s, t match. Therefore, t must also have a pair of symbols,

$$s_{k-1} = s_k \Rightarrow s_k = t_{k-1} = t_k$$

And, that shows that t is in the complement of Σ_a.

(b) Is Σ_a dense in Σ_3?

No. For example, what is the closest distance to $(22222222\cdots)$? The point in Σ_a could match the first symbol, but not the second (or, more precisely, not the first two). Therefore, the distance to the next closest point will be greater than $1/3^1 = 1/3$.

(c) Is there a point in Σ_a whose orbit under σ is dense (in Σ_a)?

As we did before, we can list all the strings of length 1, all the strings of length 2, all the strings of length 3, etc. We do have to be careful when we concatenate them, that we do not double up symbols. For example:

$$0 \;\; 1 \;\; 2 \;\; 01 \;\; 21 \;\; 02 \;\; 10 \;\; 20 \;\; 12 \;\; \ldots$$

12. If f is continuous, and A is a closed interval, show that if $f(A) \supset A$, then A contains a fixed point of f.

Let $A = [a, b]$. Suppose $f(a) < a$. Since $f(A) \supset A$, there must be a point c so that $f(c) > c$ otherwise, $f(x) < b$ for all x in $[a, b]$ and $f(A)$ would only contain $[a, b)$. Set $g(x) = f(x) - x$. Then $g(a) = f(a) - a < 0$ and $g(c) = f(c) - c > 0$, so by the Intermediate Value Theorem, there is a point t between a and c so that $g(t) = f(t) - t = 0$ (and that is the fixed point).

Alternatively, if $f(a) > a$, then there must be a point c in $[a, b]$ so that $f(c) = a < c$, because otherwise $f(A)$ would not contain a. The rest of the argument is the same as before.

13. Consider the following subsets of Σ_2 (the space of all symbol strings containing either 0 or 1). Is either dense in Σ_2?

    (a) $T_1 = \{(s_0 s_1 s_2 s_3 0 s_5 s_6 \cdots)\}$

    (b) $T_3 = \{(s_0 s_1 s_2 \cdots) \mid \text{ The sequence ends in all } 0's\}$

    Note: For more, see pg 131

    SOLUTION: These problems are here to help us with the definition of dense sets. Recall that a set A is dense in a set X if, for any point $x \in X$, either $x \in A$ or there is a sequence in A that converges to X (alternatively, there is a point in A that is within ϵ of x, for an arbitrarily small ϵ).

    In the first case, if we choose

$$x = (s_0 s_1 s_2 s_3 1 s_5 s_6 \cdots)$$

    then $x \notin T_1$, and the closest element of T_1 is when all the symbols match x except for $s_4 = 1$. Thus, the distance to T_1 is always greater than (or equal to) $1/2^4$, and T_1 is not dense in Σ.

    On the other hand, T_3 will be dense in Σ- Let x be any point in Σ,

$$x = (s_0 s_1 s_2 \cdots)$$

    Then, if x ends in all zeros, then $x \in \Sigma$. If not, the following sequence in T_3 converges to x:

$$\begin{aligned} t_1 &= (s_0 0000000 \cdots) \\ t_2 &= (s_0 s_1 000000 \cdots) \\ t_3 &= (s_0 s_1 s_2 000000 \cdots) \\ &\vdots \\ t_n &= (s_0 s_1 \cdots s_{n-1} 00000 \cdots) \end{aligned}$$

    Therefore, T_3 is dense in Σ.

14. Let Σ' be the set of all symbol strings of zeros and ones so that 0 is always followed by 1. Show that σ on Σ' is chaotic.

    Note: This is similar to Exercise 9, p 152. We want to show the three parts, (1) the periodic points are dense, (2) the system is transitive, and (3) there is sensitive dependence on initial conditions:

    - Periodic points are dense: Let $s \in \Sigma'$ be given. Show that there is a sequence of periodic points that converge to s (or show that there is a periodic point within ϵ of s):

      Define $s = (s_0 s_1 s_2 \ldots) \in \Sigma'$. Let $\epsilon > 0$ be given. Then we can find k so that $1/2^k < \epsilon$. Define a periodic point (within ϵ of s) by:

      $$t = (\overline{s_0 s_1 \ldots s_k})$$

      There is an important technicality to consider: If $s_0 = 0$ and $s_k = 0$. In this instance, take

      $$t = (\overline{s_0 s_1 \ldots s_k s_{k+1}})$$

      for if $s_k = 0$, then $s_{k+1} = 1$, and this is a valid periodic point.

    - The easiest way to show that a dynamical system is transitive is to show that there is a point (in this case, in Σ') so that the orbit of that point is dense in Σ'. We have constructed similar points before (once towards the bottom of page 116, and again in these exercises, see Exercise 11(c)). We build up this point in a similar way- Again, just be sure that you end up with a point that does not double up zeros:

      $$s = 0\ 1\ 01\ 10\ 11\ 011\ 010\ 101\ \cdots$$

      In this case, as the others, this string contains all valid strings of any finite length. Therefore, the orbit is dense.

      Be sure you understand how this shows that the system is transitive

    - Sensitive dependence on initial conditions: If s, t are different symbol strings, there exists k so that $s_k \neq t_k$. At the k^{th} iterate, $d(\sigma^k(s), \sigma^k(t)) \geq 1/2^0 = 1$. (See page 119 for more details).

15. Show that $\sigma : \Sigma \to \Sigma$ is continuous (using our usual metric d on Σ)

    See bottom of page 105 It is basically just a consequence of the Proximity Theorem.

16. Let $F : \Sigma \to \Sigma$ so that

    $$F(s_0 s_1 s_2 s_3 \cdots) = (s_0 s_2 s_4 s_5 \cdots)$$

    Is F continuous on Σ?

Error: This should be Exercise 18(f) in Chapter 9:

$$F(s_0 s_1 s_2 s_3 \cdots) = (s_0 s_2 s_4 s_6 \cdots)$$

SOLUTION:

Main Idea: To show continuity, we start with an $\epsilon > 0$, and say that $d(F(s), F(t)) < \epsilon$. We want to show that we can find $\delta > 0$ (usually as a function of ϵ), so that, if s, t are within δ of each other, then $f(s), f(t)$ must be within ϵ of each other.

We start by working backwards: Let $\epsilon > 0$ be given, and let s be given. We will show that F is continuous at s.

We can find k so that:

$$d(F(s), F(t)) < \frac{1}{2^k} < \epsilon$$

By the Proximity Theorem, the first $k+1$ symbols of $F(s)$ and $F(t)$ are the same:

$$s_0 = t_0 \quad s_2 = t_2, \quad s_4 = t_4, \ldots, \quad s_? = t_?$$

The last term is the important one, since it tells us how many symbols of the original string must be the same. For example, if $k = 0$, then we only have $s_0 = t_0$. If $k = 1$, we have the previous, and $s_2 = t_2$. If $k = 2$, then we also have $s_4 = t_4$, and $k = 3$ means we also have $s_6 = t_6$. That means, with a generic k, we have terms up to $s_{2k} = t_{2k}$.

CONCLUSION: If $\epsilon > 0$ is given, determine k so that $\epsilon < \frac{1}{2^k}$. Let $\delta = \frac{1}{2^{2k}}$.

Then:

$$d(s, t) < \delta \quad \Rightarrow \quad s_j = t_j \text{ for } j = 1, 2, \ldots, 2k$$

In particular, all the even terms are the same:

$$s_0 = t_0, s_2 = t_2, \ldots, s_{2k} = t_{2k}$$

Which means that

$$d(F(s), F(t)) < \frac{1}{2^k} < \epsilon$$

And therefore, F is continuous at any $s \in \Sigma$.