Homework Set 3 SOLUTIONS

1. The following sequences were given in the first homework set, and were found to converge for certain starting conditions. Interpret each as $a_{n+1} = F(a_n)$, and find the fixed points for F in each case, and whether or not they are attracting or repelling.

   (a) $a_{n+1} = 1/(1 + a_n)$

   $$f(x) = x \quad \Rightarrow \quad \frac{1}{1+x} = x \quad \Rightarrow \quad x = \frac{-1 \pm \sqrt{5}}{2} \doteq p_{+,-}$$

   Differentiating, we get $f'(x) = -(1+x)^{-2}$, and evaluating, we see that

   $$f'\left(\frac{-1-\sqrt{5}}{2}\right) \approx -2.6 \qquad f'\left(\frac{-1+\sqrt{5}}{2}\right) \approx -0.38$$

   Therefore, the first fixed point was repelling, the second was attracting (that was also what we found in Homework Set 1).

   (b) $a_{n+1} = \frac{1}{2}(a_n + 6)$

   $$f(x) = \frac{1}{2}(x+6) \quad \Rightarrow \quad x + 6 = 2x \quad \Rightarrow \quad x = 6$$

   Differentiating, we get $f'(x) = \frac{1}{2}$, so (every) fixed point is attracting.

   (c) $a_{n+1} = \frac{1}{2}\left(a_n + \frac{c}{a_n}\right)$, where $c > 0$ (c is fixed- Your limit will depend on c).

   $$f(x) = \frac{1}{2}\left(x + \frac{c}{x}\right) \quad \Rightarrow \quad \frac{x^2 + c}{x} = 2x \quad \Rightarrow \quad x^2 + c = 2x^2$$

   $$x = \pm\sqrt{c}$$

   Differentiating, we get $f'(x) = \frac{1}{2}\left(1 - \frac{c}{x^2}\right)$, and evaluating, we see that the derivative at either fixed point is zero- So they are both attracting (Note that the function is not defined at the origin).

2. Exercise 4(d,g) on page 50. To get a plot in Matlab, use the commands:

```
x=linspace(-4,4);  %Creates an array for the domain
yd=sin(x);  %For problem d
yg=x-x.^3;  %For problem g- Be sure to use .^
figure(1)
plot(x,yd);  %Use:  plot(x,x,x,yd); to also plot y=x
figure(2)
plot(x,yg);  %Use: plot(x,x,x,yg); to also plot y=x
```

SOLUTION: We have the cobweb function we can use to help us determine if the points are weakly attracting, repelling, or neutral. In fact, you might notice that these functions are very similar close to the fixed point at $x = 0$ (and both are weakly attracting).

3. Exercises 1(a,b,j,k) on p. 67. Use Matlab for the sketches and describe the bifurcation (Two scripts are online- One does 1(c,d), the other does 1(e,f)- These are `Ch6HW1.m` and `Ch6HW1e.m`

SOLUTIONS: We're looking for some graphical evidence of the bifurcations.

1(a) $f(x) = x + x^2 + c$ where $c = 0$. In Figure 1 (read from left to right, top to bottom) we see the graph of f as c changes from $c = -1$ to $c = 0$ to $c = 0.5$ to $c = 1$. This is pretty clearly a saddle-node bifurcation (aka blue-sky or tangent).

1(b) $f(x) = x + x^2 + c$, where $c = -1$. In Figure 2, we see a cobweb diagram (with $c = -1.2$, a little lower than the bifurcation) with an attracting two-cycle. Geometrically, we see that, at the fixed point (with $c = -1$), the slope of the tangent line would be -1, which is very suggestive of a period-doubling bifurcation.

To verify this bifurcation, we could either do a cobweb diagram (like the one shown), or show a plot of $f^2(x)$ to see the appearance of two period two points.

1(j) $f(x) = x + cx^2$ where $c = 0$. Changing c from -1 to 1, we see that we begin with one fixed point (the parabola is downward facing). At $c = 0$, we have an infinite number of fixed points (they are all fixed), then we are back to one fixed point again (the parabola is upward facing).

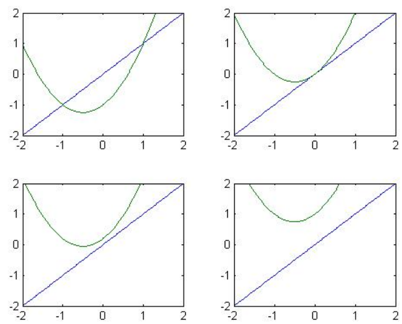

Figure 1: The graph of $f(x) = x + x^2 + c$, as c changes from -1 to 1 (read from top to bottom, left to right). This is a pretty typical picture of a saddle node bifurcation (aka blue sky or tangent).

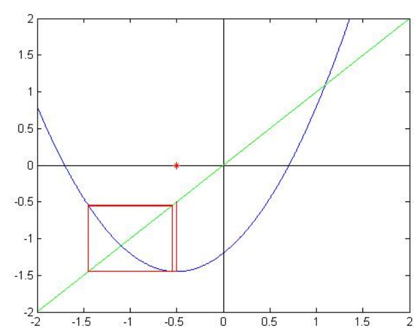

Figure 2: Figure 2: Suggests a period double bifurcation when the slope of the tangent line (at a fixed point) is -1.

Alternative: $x + cx^2 = x$ gives $cx^2 = 0$.

1(k) $f(x) = x + cx^2 + x^3$, $c = 0$.
We notice that $x = x + cx^2 + x^3$ gives

$$x^2(c + x) = 0$$

And checking the derivative, we see that the fixed point $x = 0$ is neutral, while the other one, $c = -x$, is always repelling.

4. Exercises 6-14, p. 67. Work beforehand to find the fixed points and stability:

$$\lambda x(1 - x) = x \quad \Rightarrow \quad x = 0 \text{ or } x = \frac{\lambda - 1}{\lambda}$$

Note that in this case, $\lambda \neq 0$ (We may have to consider this separately below).

Evaluating each at the derivative $f'(x) = \lambda - 2\lambda x$, we have

$$f'(0) = \lambda \qquad f'\left(\frac{\lambda - 1}{\lambda}\right) = 2 - \lambda$$

Now continue with the exercises:

6. To have an attracting fixed point at $x = 0$, the size of the derivative must be less than 1. We found that $f'(0) = \lambda$, so $-1 < \lambda < 1$.
7. For the other fixed point to be attracting, we solve for the values of λ that give the size of its derivative less than 1:

$$-1 < \lambda - 2 < 1 \quad \Rightarrow \quad 1 < \lambda < 3$$

8. Why is there a bifurcation at $\lambda = 1$? In checking our fixed points, we realize that for $0 < \lambda < 1$, we have two fixed points- one is attracting ($x = 0$) and one is repelling. At $\lambda = 1$, we have exactly one fixed point ($x = 0$), and it is neutral. For $1 < \lambda < 3$, the fixed points have swapped stability, $x = 0$ is now repelling and $x = 1 - 1/\lambda$ is attracting.

   We did not have a name for this type of bifurcation.
9. For the sketch, use Matlab (or Maple, or a graphing calculator).

10. A period-doubling bifurcation occurs at $\lambda = 3$.

    We do not see anything so far except for the loss of stability of our attracting fixed point- And the fact that the derivative here is -1. That suggests the period-doubling bifurcation.

    It is possible to find the period 2 points analytically, since we know that $\lambda x(1-x) - x$ divides the equation we get exactly, See the last problem (remember to divide polys).

11. (Use Matlab to get a sketch)

12.-13. For $-1 < \lambda < 0$, we have one attracting, one repelling fixed point. At $\lambda = -1$, the slope of the tangent line at the fixed point is -1, which would suggest a period doubling bifurcation at $x = 0$.

14. Main idea: Use long division to get a quadratic equation, which is easy to solve. In this case, we need to divide, which looks very messy but is straightforward (after a few cups of coffee).

    Initially, the division will be:

$$\frac{-\lambda^3 x^4 + 2\lambda^3 x^3 - (\lambda^2 + \lambda^3)x^2 - (1-\lambda^2)x}{-\lambda x^2 - (1-\lambda)x}$$

Take out the common factor of $-x$ to do the division below:

$$\begin{array}{r l}
 & \lambda^2 x^2 - \lambda(1+\lambda)x + (1+\lambda) \\
\lambda x + (1-\lambda) \,\big) & \lambda^3 x^3 - 2\lambda^3 x^2 \qquad + \lambda^2(1+\lambda)x + (1-\lambda^2) \\
 & \underline{-\lambda^3 x^3 + \lambda^2(1-\lambda)x^2} \\
 & \qquad -\lambda^2(1+\lambda)x^2 + \lambda^2(1+\lambda)x \\
 & \qquad \underline{-\lambda^2(1+\lambda)x^2 - \lambda(1-\lambda^2)x} \\
 & \qquad\qquad\qquad\qquad \lambda(\lambda+1)x + (1-\lambda^2)
\end{array}$$