Notes about 9.4

The Quadratic Mapping, $Q_c(x)$

$Q_c(x) = x^2 + c$ is the quadratic map. When $c < -2$, we saw that all the interesting behavior occurs on the interval I, which is $[-p_+, p_+]$ (the value of p_+ was found earlier- It is the rightmost fixed point of Q_c).

We constructed a set similar to the Cantor set on I by removing open intervals,

$$A_i = \left\{ x \in I \,|\, Q_c^i(x) < -p_+ \right\}$$

The set that remained was called Λ. This is the set of points that remain in I *for all* iterations n.

Symbol Space

Define the space of symbols,

$$\Sigma = \{s_0 s_1 s_2 \ldots \,|\, s_j = 0 \text{ or } 1\}$$

We define a mapping $\sigma : \Sigma \to \Sigma$,

$$\sigma(s) = \sigma(s_0 s_1 s_2 s_3 \ldots) = s_1 s_2 s_3 \ldots$$

The action of σ is very simple: It just drops the first symbol off of the end of the symbol string. In order to look at the geometry of Σ, we need a way of measuring distance between things:

$$d(s,t) = \sum_{j=0}^{\infty} \frac{|s_j - t_j|}{2^j}$$

We saw that the distance between two things in Σ, say s, t, are within $1/2^n$ if and only if the first $n+1$ symbols in each are the same: $s_0 = t_0, s_1 = t_1, \cdots, s_n = t_n$.

A strong relationship

What are we up to in Chapter 9? We are building up the tools that tie the seemingly very complicated dynamical system:

$$Q_c(x) : \Lambda \to \Lambda$$

to the very simple dynamical system:

$$\sigma : \Sigma \to \Sigma$$

in the sense that every orbit in the Quadratic map dynamical system has a corresponding orbit in the symbol space dynamical system, and vice-versa. Thus, the action of Q_c, while looking very complicated, can be understood by just looking at σ acting on Σ.

How can we tie the dynamical systems together? Through a special type of function called a homeomorphism. Our goal will be to show that the itinerary map,

$$S(x) = \left\{ s_0 s_1 s_2 \ldots \,|\, x \in I_{s_0},\ Q_c(x) \in I_{s_1},\ Q_c^2(x) \in I_{s_2}, \ldots \right\}$$

is the homeomorphism (we will say that Λ and Σ are therefore homeomorphic).

Before we get too specific, let us back up and look at some exercises that will help us recall things that are true about functions in general.

Definitions

- A function is $1-1$ if, for every x, y in the domain,

  $$x \neq y \Rightarrow f(x) \neq f(y) \text{ or equivalently } f(x) = f(y) \Rightarrow x = y$$

  Graphically, a function is $1-1$ if it passes the "horizontal line test".

- A function $f : X \to Y$ is "onto" if, for every $y \in Y$, there is an $x \in X$ such that $f(x) = y$.

- The "image" of a set A under the mapping f:

  $$f(A) = \{y \,|\, f(x) = y \text{ for } x \in A\}$$

  Notice that every function is "onto" the image of the domain. The image is always a subset of the range.

- The "preimage" of a set B under the mapping f:

  $$f^{-1}(B) = \{x \,|\, f(x) \in B\}$$

  This notation may be confusing - We are **not** implying that the function f has an inverse. The preimage of a set can always be found, whether f has an inverse or not. The preimage is always a subset of the domain.

- A function f is continuous at $x = a$ if, for every $\epsilon > 0$, there is a $\delta > 0$ such that,

  $$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

  To show that a function is continuous at a certain point, you assume that an arbitrary ϵ has been given. You then need to construct a δ (typically as an expression in ϵ) so that the above implication is true.

- A function f is called a **homeomorphism** if it is 1-1, onto, continuous and f^{-1} is continuous. If $f : X \to Y$ is a homeomorphism, X and Y are said to be **homeomorphic**.

Exercises with the Definitions

1. What is the difference between the "range", the "codomain", and the "image of the domain"?

2. Let $f(x) = 1 - 2x$. Show that f is $1-1$ and onto the real numbers, and show that it is continuous at $x = 1$.

3. Let $f(x) = 1 - x^2$.

   (a) Show (graphically and algebraically) that f is not 1-1.

   (b) Find the image of the interval $[-1, 2]$.

   (c) Find the preimage of the interval $[1/4, 3/4]$.

   (d) Find intervals A so that $f^2(A) = [1/4, 3/4]$.

   (e) Let $A = [0, 2]$ and $B = [-1, 1]$. Show graphically what the following equation means (and algebraically, what is it?)

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$$

4. Show, by means of an example, that if f is not continuous, then the image of a closed interval could be either closed or open (an open interval is one that does not contain its endpoints) or neither.

5. As in our text, define

$$I_{s_0 s_1} = \{x \in I \,|\, x \in I_{s_0}, Q_c(x) \in I_{s_1}\}$$

   Show graphically (along the line $y = x$) where I_{01} is, and show that

$$I_{01} = I_0 \cap Q_c^{-1}(I_1)$$

6. Continuing the last problem, show that:

$$I_{010} = I_0 \cap Q_c^{-1}(I_{10})$$

7. (Same notation as last two problems) Show that

$$I_{010} \subset I_{01} \subset I_0$$

8. Let g be defined piecewise as:

$$g(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ x - 1 & \text{if } 2 < x \le 3 \end{cases}$$

   Define $g : [0, 1] \cup (2, 3] \to [0, 2]$.

   (a) Show that g is $1-1$ and onto (which makes g invertible).

   (b) Find g^{-1} and conclude that g is continuous, but g^{-1} is not.