Math 124 Review Weeks 1-8

Information about the exam.

The exam will be 50 minutes in length. You may have one-half sheet of paper (half of 8.5×11 inches, or its equivalent) of notes written on one side. You may use a calculator, but **only for numerical calculations- no graphs**.

Please note that even with notes and a calculator, you'll still need to study for the exam, as you will not have enough time to approach every problem from scratch- Hopefully every problem on the exam will be similar to something you've seen.

Please read the instructions carefully. A couple of examples:

- If an **exact value** is requested, a **numerical approximation** may not give you full points (for example, $\sqrt{3}/2$, $\pi/4$ are exact values, as is something like $\log_2(3)$).
- If a limit is to be constructed algebraically, a numerical table will not give full points.
- If a derivative using the definition is called for, using a shortcut formula (which we have not learned in class yet) may not give you any points.

Discussion

In this portion of the course, we have focused on building up the tools surrounding the building and analyzing of functions, and we review what we might call a toolbox of functions. To perform the analysis in calculus, we'll also need to use the tools of algebra (to do things like factoring, working with fractions, solve basic equations for an unknown variable, and so on).

We have also started discussing three key ideas from calculus: The limit, continuity, and the instantaneous rate of change.

The primary tools for general functions

1. Know the definition of a function, and an inverse function.

   We should know when a given relation defines a function, and how to determine if the inverse is (or will be) a function itself. Be able to find an inverse graphically, numerically, and algebraically.

   We also discussed piecewise defined functions generally (and focused on the absolute value function).

2. Function notation, in particular function composition.

   Given particular functions $f(x), g(x)$ be able to construct $f \circ g$. In reverse, given a function $h(x)$, be able to construct f, g so that $h(x) = f \circ g$.

   Given a function $f(x)$, be able to compute expressions like:

   $$\frac{f(a+h) - f(a)}{h} \qquad \text{or} \qquad \frac{f(x+h) - f(x)}{h}$$

   Understand that these expressions are computing the **average rate of change** of a given function, or in our new notation, $AV_{[a,a+h]}$.

3. The limit of a given function, $f(x)$.

   $$\lim_{x \to a^+} f(x), \qquad \lim_{x \to a^-} f(x), \qquad \lim_{x \to a} f(x)$$

   Be able to define what it means if $\lim_{x \to a} f(x) = L$.

Be able to compute the limit of a given function graphically and algebraically. For the algebra, we might need to **factor and cancel**, or **multiply by the conjugate** (examples in the review below). Given a piecewise defined function, you may need to determine which parts of the function to use for a given limit.

4. The instantaneous rate of change of a function (the derivative):

   There is a distinction between the two **definitions of the derivative** of f (**don't forget the limit!**):

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \qquad \text{versus} \qquad f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

   On the left, $f'(a)$ is a **number** giving the instantaneous rate of change of f at $x = a$. This is also interpreted as the **slope of the tangent line** to $f(x)$ at the point $(a, f(a))$.

   On the right, $f'(x)$ is a **function** that gives you a **formula** you can use to compute the instantaneous rate of change of f at any x (in its domain).

   Remember, there are a two distinct kinds of limits we've focused on- Understand the difference between the following:

$$\lim_{x\to a} f(x) \qquad \text{versus} \qquad \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

   For a given function, we should be able to estimate $f'(a)$ graphically, or compute $f'(a)$ numerically or algebraically. Similarly, given $f(x)$ be able to sketch a graph of $f'(x)$.

   Be able to construct the tangent line to $f(x)$ at $x = a$ (that is the line through the point $(a, f(a))$ with slope $f'(a)$, or $y - f(a) = f'(a)(x-a)$, which we also called $L(x) = f(a) + f'(a)(x-a)$.

   We also discussed what the value of f' (or $f''(x)$) tells us about the graph of f:

   - If $f'(a) > 0$, then f is increasing at $x = a$.
   - If $f'(a) < 0$, then f is decreasing at $x = a$.
   - If $f''(a) > 0$, then f is concave up at $x = a$.
   - If $f''(a) < 0$, then f is concave down at $x = a$.

   Given the graph of $f'(x)$, discuss properties of $f(x)$ and $f''(x)$.

5. Continuity and Differentiability.

   - Definition: $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.

     Know the definition, and understand that it means that there are three things that must be true in this statement (the limit exists- left and right, $f(a)$ exists, and these two things must be the same).

     Comment: The idea that f is continuous if you don't need to raise your pencil from the page when drawing its graph is a nice intuitive idea, but this is NOT the definition.
   - Definition: $f(x)$ is differentiable at $x = a$ if $f'(a)$ exists.

     Graphically determine when f is differentiable (no sharp points on the graph, no vertical asymptotes, and f must also be continuous- see the next item).
   - Know the relationship between differentiability and continuity: A function can be continuous at $x = a$ and not differentiable there (classic example: $f(x) = |x|$ at $a = 0$). However, if a function is differentiable at $x = a$, it is automatically also continuous there (we say that differentiability is a **stronger** condition that continuity).

The toolbox of functions

1. Lines

   Know what a line is (constant rate of change). Be able to construct the equation of a line given either two points, or a point and a slope. Be able to estimate the slope of a line segment (important for the derivative).

2. Absoute value

   Know what a piecewise defined function is, and how the absolute value function is defined. Be able to use the definition to analyze the behavior of functions like $|x-3|/(x-3)$, for example.

3. Quadratics

   Be able to factor "simple" quadratics, especially using the difference of squares formula. Be able to solve a quadratic equation using the quadratic formula.

   We looked at some physics using quadratics (a ball thrown in the air). We had two different equations that gave us different information:

   $$f(t) = -16t^2 + v_0 t + h_0 \qquad f(t) = a(t-h)^2 + k$$

   In the first equation, we can read off initial velocity and height. From the second equation, the position of the vertex is at (h, k). (Side note: The number -16 comes from acceleration due to gravity, which is estimated to be 32 feet/sec^2, but that's not something you need to know for the exam).

4. Exponential functions

   Be able to model population and the law of cooling using exponential functions (we've talked about things like doubling time, and using $f(t) = a \cdot b^t + c$ in these models). Be able to sketch the graph of an exponential function.

   Be able to apply the rules of exponents.

5. Logarithmic functions

   Be able to define the logarithm as the inverse of the exponential function.

   Be able to apply the rules of logarithms.

6. Trig functions (focus on sine, cosine and tangent)

   We had two sections on trig: The unit circle and right triangle trig. Be able to define and use sine, cosine and the tangent function using either. Know the two special triangles (45-45-90 and 30-60-90) and be able to construct the unit circle (with special angles) in the first quadrant. Be able to find the trig function values at other special angles on the unit circle.

   Be able to sketch the sine, cosine and tangent functions.

   Discuss why we have to restrict the domain on the sine, cosine and tangent functions to invert them, and know what those standard restrictions are. Be able to sketch the inverse tangent function.

   Be able to use the inverse sine, cosine and tangent functions when solving an equation involving sines, cosines and the tangent function

Review Questions

This is not meant to be an exhaustive list of questions, but just a list "out of context" to help you study. If something is missing from this list, don't assume that it won't show up on the exam (and everything below may not be on the exam). The exam will be approximately half on precalculus topics and about half on limits, continuity and the derivatives.

1. Finish each definition:
   (a) $\lim_{x \to a} f(x) = L$ means that:
   (b) A function f is continuous at $x = a$ if:
   (c) The derivative of $f(x)$ is:
2. (a) Given $f(x) = 3x + 2$ and $g(x) = x^2 + x + 1$, then find expressions for $f \circ g$ and $g \circ f$.
   (b) If $h(x) = \sqrt{3x^2 + 2x}$, find f, g so that $h(x) = f(g(x))$.
3. Given a table of (x, y) data points, how would we check to see if the data represents a **line**?
4. Consider the exponential function:

   $$Q(t) = 30.8(0.751)^t.$$

   Give the starting value a, the growth factor b and the growth rate r if $Q = ab^t = a(1 + r)^t$.
5. A population has 8000 at time $t = 0$ (t in years).
   (a) If the population decreases by 125 people per year, find a formula for the population P at time t.
   (b) If the population decreases by 6% per year, find a formula for the population P at time t.
   (c) If the initial population is doubled in 3 years, find a formula for the population P at time t.
6. If $F(t) = -4e^{-2t} + 50$, what is the long term behavior of F as $t \to \infty$?
7. Find the exact value of each variable:
   (a) $e^{2\ln(t)} = 4$
   (b) $3 \cdot 10^t + 11 = 101$
   (c) $2\log_3(y) + 5 = 1$
   (d) $19 - \ln(3 - x) = 0$
   (e) $11 + 5^{5x} = 16$
   (f) $\log_5(5^{2x+1}) = 2$
8. If a right triangle has one angle that is $\theta = 40$ degrees, and the length of the side opposite is 4, find all remaining angles and the lengths of the remaining sides exactly (not as numerical approximations).
9. Consider the parabola $f(x) = -2x^2 + 4x + 5$
   (a) Find the $x-$ and $y-$ coordinates of the vertex.
   (b) What is the $y-$intercept of the parabola?
   (c) Is the parabola concave up or concave down? (How do you know?)
   (d) Plot the points where $x = 0$, $x = 1$, $x = 3$.
   (e) Find $AV_{[0,1]}$, $AV_{[1,2]}$, $AV_{[2,3]}$ and use the results to infer $AV_{[3,4]}$.
10. A cup of coffee at 180 degrees (F) is placed in a room with temperature of 72 degrees (F). After 10 minutes, the temperature of the coffee is 135 degrees. Use the model for heating and cooling (Newton's law of cooling) to determine when the temperature is 100 degrees (F).
11. A person is standing 50 feet from a streetlight observes that they cast a shadow that is 10 feet long. If a ray of light from the streetlight to the tip of the person's shadow forms an angle of 26 degrees with the ground, how tall is the person and how tall is the streetlight? You may use numerical approximations.
12. Find exact values of the following (no numerical approximations), if possible.

(a) $\sin(4\pi/3)$

(b) $\sec(3\pi/4)$

(c) $\cot(5\pi/3)$

(d) $\arcsin(1/2)$

(e) $\arctan(1)$

(f) $\arccos(2)$

(g) $\arcsin(\sin(\pi/3))$

(h) $\cos(\arcsin(-\sqrt{3}/2))$

(i) $\arctan(\tan(7\pi/4))$

13. In a right triangle with hypotenuse 1 and vertical leg x, with angle θ opposite x, determine the simplest expression you can for each of the following quantities in terms of x:

(a) $\sin(\theta) =$ (b) $\sec(\theta) =$ (c) $\cos(\arcsin(x)) =$ (d) $\cot(\arcsin(x)) =$

14. If $f(x)$ is as given below, at what value of c is the function f continuous at every point?

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x < 3 \\ 2cx & \text{if } x \geq 3 \end{cases}$$

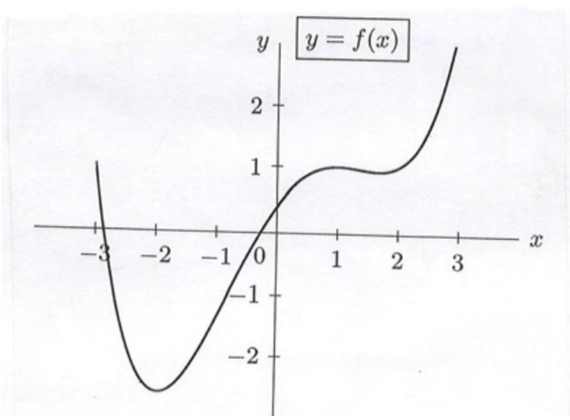
15. For the function

$$f(x) = \begin{cases} x^2 - 4 & \text{if } 0 \leq x < 4 \\ 4 & \text{if } x = 4 \\ 3x + 2 & \text{if } x < 4 \end{cases}$$

Use algebra to compute each of the limits (or enter DNE if the limit does not exist).

(a) $\lim_{x\to 4^+} f(x) =$ (b) $\lim_{x\to 4^-} f(x) =$ (c) $\lim_{x\to 4} f(x) =$ (d) $\lim_{x\to 2} f(x) =$

16. Given the graph of $y = f(x)$ below, next to it sketch a graph of $f'(x)$.



17. Find the value of each limit algebraically (you may check your work numerically, but you should support it with your algebra):

(a) $\lim_{x\to 4} \frac{x^2-16}{x-4}$ (b) $\lim_{x\to 3^-} \frac{|x-3|}{x-3}$ (c) $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$

18. For each function, use the definition of $f'(a)$ to compute the derivative of the given point at $a = 1$.

(a) $f(x) = 3x^2$ (b) $f(x) = 1/x$ (c) $f(x) = \sqrt{x}$

19. For each function, determine $f'(x)$ using the definition of the derivative:

(a) $f(x) = x^2 - 3x$ (b) $f(x) = 1/x$ (c) $f(x) = |x - 1|$ (Note *)

(*) For part (c) , you may use graphical reasoning.

20. If $f(1) = 2$ and $f'(1) = -3$, use a linear approximation (tangent line approximation) to approximate $f(1.3)$.

21. Suppose that $y = f(x)$ is a twice differentiable function such that f'' is continuous, and we know the following information:

$$f(2) = -3 \qquad f'(2) = 1.5 \qquad f''(2) = -0.25$$

   (a) Is f increasing or decreasing near $x = 2$? Is f concave up or concave down near $x = 2$?

   (b) Do you expect $f(2.1)$ to be:

      i. Greater than -3 ii. Equal to -3 iii. Less than -3

   (c) Do you expect $f'(2.1)$ to be:

      i. Greater than 1.5 ii. Equal to 1.5 iii. Less than 1.5

   (d) Find the equation of the tangent line to $f(x)$ at $x = 2$.

   (e) Sketch a graph of $f(x)$ near the point $(2, f(2))$.

22. For each of the following prompts, give an example of a function that satisfies the stated criteria. A formula or a graph is sufficient for each. If no such example is possible, explain why.

   (a) A function f that is continuous at $a = 2$, but is not differentiable at $a = 2$.

   (b) A function g that is differentiable at $a = 3$, but does not have a limit at $a = 3$.

   (c) A function h that has a limit at $a = -2$, is defined at $a = -2$, but is not continuous at $a = -2$.

   (d) A function p that satisfies all of the following:

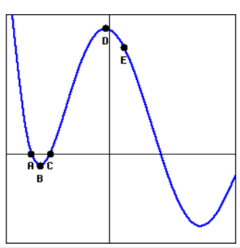
      i. $p(-1) = 3$, $\lim_{x \to -1} p(x) = 2$

      ii. $p(0) = 1$ and $p'(0) = 0$

      iii. $\lim_{x \to 1} p(x) = p(1)$

      iv. $p'(1)$ does not exist.

23. The graph below is of a function $y = f(x)$, and 5 points are shown. Select the correct signs for each of f, f' and f'' at each point. If the graph is too small, see Exercise 2 of Section 1.6 (AC).



Point	A	B	C	D	E
f	○ positive ○ zero ○ negative	○ positive ○ zero ○ negative	○ positive ○ zero ○ negative	○ positive ○ zero ○ negative	○ positive ○ zero ○ negative
f'	○ positive ○ zero ○ negative	○ positive ○ zero ○ negative	○ positive ○ zero ○ negative	○ positive ○ zero ○ negative	○ positive ○ zero ○ negative
f''	○ positive ○ zero ○ negative	○ positive ○ zero ○ negative	○ positive ○ zero ○ negative	○ positive ○ zero ○ negative	○ positive ○ zero ○ negative