Solutions to the Review

1. Finish each definition:

   (a) $\lim_{x\to a} f(x) = L$ means that: we can make $f(x)$ arbitrarily close to L by making x sufficiently close to a.

   (b) A function f is continuous at $x = a$ if: $\lim\limits_{x\to a} f(x) = f(a)$.

   Also know that this definition implies that three things are true: The limit exists at $x = a$, the function exists at $x = a$, and these two things are the same.

   (c) The derivative of $f(x)$ is: $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

   Also know that this is a formula for the slope of the tangent line to f at x, and represents a formula for the instantaneous rate of change of f at x.

2. (a) Given $f(x) = 3x + 2$ and $g(x) = x^2 + x + 1$, then find expressions for $f \circ g$ and $g \circ f$.

   SOLUTION:

$$f \circ g = f(g(x)) = f(x^2 + x + 1) = 3(x^2 + x + 1) + 2$$
$$g \circ f = g(3x + 2) = (3x + 2)^2 + (3x + 2) + 1$$

   (b) If $h(x) = \sqrt{3x^2 + 2x}$, find f, g so that $h(x) = f(g(x))$.

   SOLUTION: Several ways to do this, but the most obvious is probably to take

$$g(x) = 3x^2 + 2x \qquad \text{and} \qquad f(x) = \sqrt{x}$$

3. Given a table of (x, y) data points, how would we check to see if the data represents a **line**?

   SOLUTION: Data represents a line when the average rate of change (between any two points) is constant (in fact, this number is the slope of the line).

4. Consider the exponential function:

$$Q(t) = 30.8(0.751)^t.$$

   Give the starting value a, the growth factor b and the growth rate r if $Q = ab^t = a(1 + r)^t$.

   SOLUTION: The starting value is when $t = 0$, so here, $Q(0) = a \cdot b^0 = a$. The growth factor b is 0.751, (which is actually decay rather than growth), and the growth/decay rate r is 0.249, meaning that Q is loosing approximately 25% per year.

5. A population has 8000 at time $t = 0$ (t in years).

   (a) If the population decreases by 125 people per year, find a formula for the population P at time t.

   SOLUTION: This is a linear function, with a slope of -125:

$$P(t) = 8000 - 125t$$

   (b) If the population decreases by 6% per year, find a formula for the population P at time t.

$$P(t) = 8000\,(0.94)^t$$

   (c) If the initial population is doubled in 3 years, find a formula for the population P at time t.

   SOLUTION: First, we'll assume that $P(t) = Ae^{kt}$ and find the value of k.

$$P(3) = 16000 \quad \Rightarrow \quad 16000 = 8000e^{3k} \quad \Rightarrow \quad k = \ln(2)/3$$

   Then

$$P(t) = 8000e^{\ln(2)\,t/3}$$

6. If $F(t) = -4e^{-2t} + 50$, what is the long term behavior of F as $t \to \infty$?

   SOLUTION: As t gets large, $e^{-2t} = \frac{1}{e^{2t}} \to 0$, so overall, this function approaches 50.

7. Find the exact value of each variable:

   (a) $e^{2\ln(t)} = 4$
   SOLUTION: We use $a\ln(b) = \ln(b^a)$ and $e^{\ln(a)} = a$, to get

   $$e^{2\ln(t)} = e^{\ln(t^2)} = t^2$$

   so that the equation above simplifies to $t^2 = 4$, or $t = \pm 2$. We choose only $t = 2$ since the domain of the natural log (and all logs) is $t > 0$.

   (b) $3 \cdot 10^t + 11 = 101$
   SOLUTION: First solve for 10^t, then apply logs:

   $$\begin{aligned} 3 \cdot 10^t + 11 &= 101 \\ 3 \cdot 10^t &= 90 \\ 10^t &= 30 \\ t &= \log_{10}(30) \end{aligned}$$

   (c) $2\log_3(y) + 5 = 1$
   SOLUTION: First solve for $log_3(y)$, then apply the exponential:

   $$\begin{aligned} 2\log_3(y) + 5 &= 1 \\ 2\log_3(y) &= -4 \\ \log_3(y) &= -2 \\ 3^{-2} &= y \end{aligned}$$

   so that $y = 1/9$.

   (d) $19 - \ln(3 - x) = 0$
   SOLUTION: Same idea as the previous problem. Find that $x = 3 - e^{19}$.

   (e) $11 + 5^{5x} = 16$
   SOLUTION: We should find that $x = 1/5$.

   (f) $\log_5(5^{2x+1}) = 2$
   SOLUTION: We should find that $x = 1/2$.

8. If a right triangle has one angle that is $\theta = 40$ degrees, and the length of the side opposite is 4, find all remaining angles and the lengths of the remaining sides exactly (not as numerical approximations).

   SOLUTION: Draw a sketch first. The remaining angle is 50°. There are several ways to construct the legs and hypotenuse of the triangle- one way is given below:

   The two legs of the triangle can be computed as 4 and $4/\tan(40)$, and the hypotenuse can be computed as $4/\sin(40)$.

9. Consider the parabola $f(x) = -2x^2 + 4x + 5$

   (a) Find the $x-$ and $y-$ coordinates of the vertex.
   SOLLUTION: Given ax^2+bx+c, the vertex occurs at $x = -b/2a$, so in this case, the $x-$coordinate is $x = -4/2(-2) = 1$. Then the $y-$coordinate is

   $$f(1) = -2 + 4 + 5 = 7 \quad \Rightarrow \quad \text{The vertex is at } (1, 7)$$

(b) What is the $y-$intercept of the parabola?

SOLUTION: The $y-$intercept is where $x = 0$, so $f(0) = 5$.

(c) Is the parabola concave up or concave down? (How do you know?)

SOLUTION: The parabola is concave down since the leading term (the number in front of x^2) is a negative number (the parabola opens down).

(d) Plot the points where $x = 0$, $x = 1$, $x = 3$.

SOLUTION: Doe this on Desmos- The point is to get a table of values for the next question. Sorry- $x = 2$ was left off the list. Go ahead and add it in.

x	0	1	2	3
y	5	7	5	−1

(e) Find $AV_{[0,1]}$, $AV_{[1,2]}$, $AV_{[2,3]}$ and use the results to infer $AV_{[3,4]}$.

SOLUTION: Using the table of values in the last part,

$$AV_{[0,1]} = \frac{7-5}{1-0} = 2 \qquad AV_{[1,2]} = \frac{5-7}{2-1} = -2 \qquad AV_{[2,3]} = \frac{-1-5}{3-2} = -6$$

Apparently, we're dropping four units for every one unit out. We would infer that that $AV_{[3,4]} = -10$.

10. A cup of coffee at 180 degrees (F) is placed in a room with temperature of 72 degrees (F). After 10 minutes, the temperature of the coffee is 135 degrees. Use the model for heating and cooling (Newton's law of cooling) to determine when the temperature is 100 degrees (F).

SOLUTION: Our model is $F(t) = a \cdot b^t + c$, where c is the room (environmental) temperature. Using that, we expect that $a \cdot b^t \to 0$ as t gets large (so that the temp of the coffee approaches the temperature of the room).

We can use the data to solve for a, b: At time 0, the temperature is 180, so

$$180 = a \cdot b^0 + 72 \quad \Rightarrow \quad a = 180 - 72 = 108.$$

Next, at time 10 (minutes), the coffee temperature is 135. Assuming time to be in minutes,

$$135 = 108 \cdot b^{10} + 72 \quad \Rightarrow \quad \frac{7}{12} = b^{10} \quad \Rightarrow \quad b = (7/12)^{1/10} \approx 0.94753$$

Now we know that the model is given by

$$F(t) = 108 \cdot (0.94753)^t + 72$$

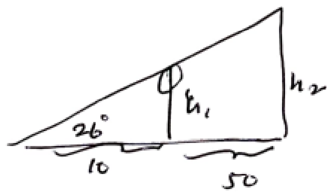
To find when the temperature is 100, solve the equation for t:

$$100 = 108 \cdot (0.93303)^t + 72 \quad \Rightarrow \quad 0.2593 = 0.94753^t \quad \Rightarrow \quad t = \frac{\ln(0.2593)}{\ln(0.94753)} \approx 25.045$$

so about 25 minutes.

11. A person is standing 50 feet from a streetlight observes that they cast a shadow that is 10 feet long. If a ray of light from the streetlight to the tip of the person's shadow forms an angle of 26 degrees with the ground, how tall is the person and how tall is the streetlight? You may use numerical approximations.

SOLUTION:



From the setup, if h_1 is the height of the person and h_2 is the height of the streetlight, we could compute them as:

$$h_1 = 10\tan(26) \approx 4.877 \qquad h_2 = 60\tan(26) \approx 29.264$$

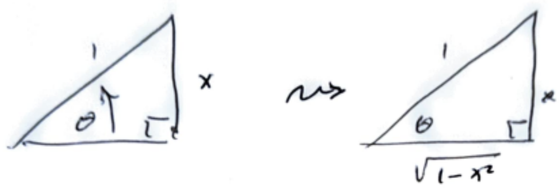
12. Find exact values of the following (no numerical approximations), if possible.

(a) $\sin(4\pi/3) = -\sqrt{3}/2$.
(b) $\sec(3\pi/4) = -\sqrt{2}$
(c) $\cot(5\pi/3) = -1/\sqrt{3}$
(d) $\arcsin(1/2) = \pi/6$
(e) $\arctan(1) = \pi/4$
(f) $\arccos(2)$ DNE (*)
(g) $\arcsin(\sin(\pi/3)) = \pi/3$
(h) $\cos(\arcsin(-\sqrt{3}/2)) = 1/2$
(i) $\arctan(\tan(7\pi/4)) = -\pi/4$

For note (*), there is no angle such that the cosine of the angle is 2 (cosine only goes between ± 1).

13. In a right triangle with hypotenuse 1 and vertical leg x, with angle θ opposite x, determine the simplest expression you can for each of the following quantities in terms of x.

SOLUTION: Using the triangle as a reference:



(a) $\sin(\theta) = x$
(b) $\sec(\theta) = \frac{\text{hyp}}{\text{adj}} = \frac{1}{\sqrt{1-x^2}}$
(c) $\cos(\arcsin(x)) = \cos(\theta) = \sqrt{1-x^2}$
(d) $\cot(\arcsin(x)) = \cot(\theta) = \frac{\text{adj}}{\text{opp}} = \frac{\sqrt{1-x^2}}{x}$

14. If $f(x)$ is as given below, at what value of c is the function f continuous at every point?

$$f(x) = \begin{cases} x^2 - 1 & \text{if } x < 3 \\ 2cx & \text{if } x \geq 3 \end{cases}$$

SOLUTION: For f to be continuous at $x = 3$, $\lim_{x\to 3} f(x) = f(3)$. First, let's compute the limit- notice that we have to compute the limit from the left and right of 3 separately:

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x^2 - 1 = 3^2 - 1 = 9 - 1 = 8$$

Next from the other side:

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 2cx = 6c$$

Furthermore, $f(3) = 6c$.

Therefore, for the function to be continuous at $x = 3$, $6c = 8$, or $c = 4/3$. The function is continuous at every other point of x, since it is either $x^2 - 1$ or $8x/3$.

15. (**TYPO** in the original function. Change the last one to read as follows below).

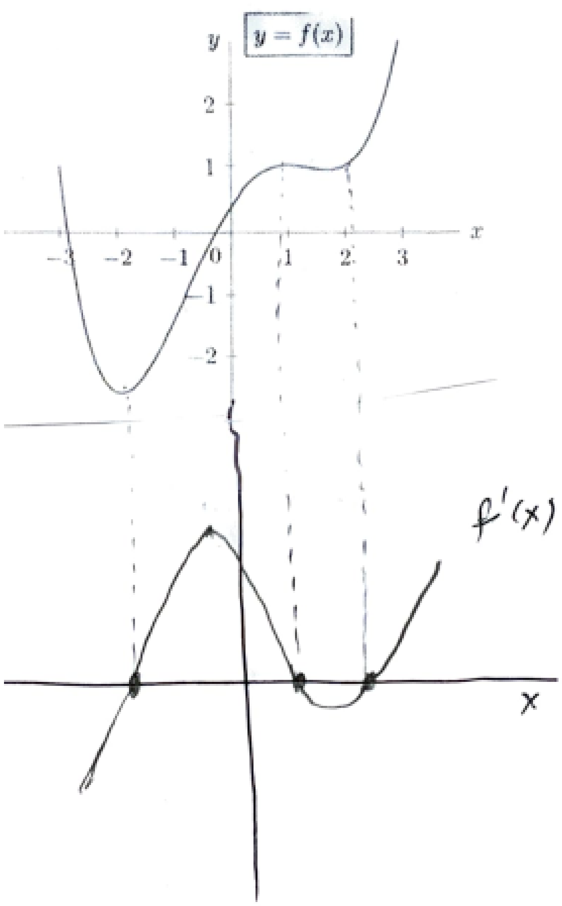
For the function

$$f(x) = \begin{cases} x^2 - 4 & \text{if } 0 \le x < 4 \\ 4 & \text{if } x = 4 \\ 3x + 2 & \text{if } x > 4 \end{cases}$$

Use algebra to compute each of the limits (or enter DNE if the limit does not exist).

(a) $\lim_{x \to 4^+} f(x) = 14$ (b) $\lim_{x \to 4^-} f(x) = 12$ (c) $\lim_{x \to 4} f(x) DNE$ (d) $\lim_{x \to 2} f(x) = 0$

16. Given the graph of $y = f(x)$ below, next to it sketch a graph of $f'(x)$.



17. Find the value of each limit algebraically (you may check your work numerically, but you should support it with your algebra):

(a) $\lim_{x \to 4} \frac{x^2 - 16}{x - 4} = \lim_{x \to 4} \frac{(x+4)(x-4)}{x - 4} = 8$

(b) $\lim_{x \to 3^-} \frac{|x - 3|}{x - 3}$

If $x < 3$, then $|x - 3|$ will be a negative number, so

$$\frac{|x-3|}{x-3} = \frac{-(x-3)}{x-3} = -1$$

Therefore, the limit as $x \to 3$ from the left is -1. (The limit from the right would be +1, by the way).

(c) $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} = \lim_{x\to 0} \frac{\sqrt{x+1}-1}{x} \cdot \frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \lim_{x\to 0} \frac{(x+1)-1}{x(\sqrt{x+1}+1)} = \frac{1}{2}$

18. For each function, use the definition of $f'(a)$ to compute the derivative of the given point at $a = 1$.

(a) $f(x) = 3x^2$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{3(1+h)^2-3}{h} = \lim_{h\to 0} \frac{(3+6h+3h^2)-3}{h} = \lim_{h\to 0} 6+3h = 6$$

(b) $f(x) = 1/x$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{\frac{1}{1+h}-1}{h} = \lim_{h\to 0} \frac{\frac{1-(1+h)}{1+h}}{h} = \lim_{h\to 0} -\frac{1}{1+h} = -1$$

(c) $f(x) = \sqrt{x}$

This one is very similar to 17(c).

$$f'(1) = \lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} = \lim_{x\to 0} \frac{\sqrt{1+h}-1}{h} \cdot \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1} = \lim_{x\to 0} \frac{(1+h)-1}{h(\sqrt{1+h}+1)} = \frac{1}{2}$$

19. For each function, determine $f'(x)$ using the definition of the derivative:

(a) $f(x) = x^2 - 3x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

At this point, it might be best to simplify each part and put them together at the end:

$$f(x+h) = (x+h)^2 - 3(x+h) = x^2 + 2xh + h^2 - 3x - 3h$$

And

$$f(x) = x^2 - 3x$$

Therefore, the numerator is

$$f(x+h) - f(x) = x^2 + 2xh + h^2 - 3x - 3h - (x^2 - 3x) = 2xh + h^2 - 3h = h(2x + h - 3)$$

Now going back to the limit:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{h(2x+h-3)}{h} = 2x - 3$$

(b) $f(x) = 1/x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

(c) $f(x) = |x-1|$

For this one, let's see how far we can get by using the derivative of a line.

First, rewrite the absolute value function:

$$f(x) = |x-1| = \begin{cases} -(x-1) & \text{if } x \le 1 \\ x-1 & \text{if } x > 1 \end{cases}$$

We have two lines joined together- Remember that the derivative of a line is the slope, so differentiating here, we get:

$$f'(x) = \begin{cases} -1 & \text{if } x < 1 \\ 1 & \text{if } x > 1 \end{cases}$$

And because f has a sharp vertex at $x = 1$, $f'(1)$ does not exist.

20. If $f(1) = 2$ and $f'(1) = -3$, use a linear approximation (tangent line approximation) to approximate $f(1.3)$.

SOLUTION: If you reason this out, we start at $x = 1$ and move to 1.3, meaning we increase x by 0.3. The derivative is -3, so therefore, we will move vertically by $(-3)(0.3) = -0.9$ starting from 2. So that puts the estimated value at 1.1.

Alternatively, using the equation of the tangent line,

$$y - 2 = -3(x-1) \quad \Rightarrow \quad y = 2 - 3(x-1)$$

Use the line for the approximation of $f(1.3)$:

$$f(1.3) \approx 2 - 3(1.3 - 1) = 2 - 0.9 = 1.1$$

21. Suppose that $y = f(x)$ is a twice differentiable function such that f'' is continuous, and we know the following information:

$$f(2) = -3 \qquad f'(2) = 1.5 \qquad f''(2) = -0.25$$

(a) Is f increasing or decreasing near $x = 2$?

ANSWER: Since $f'(2) > 0$, f is increasing near $x = 2$.

Is f concave up or concave down near $x = 2$?

ANSWER: Since $f''(2) < 0$, f is concave down near $x = 2$.

(b) We expect that $f(2.1)$ will be GREATER THAN -3, because f is increasing at $x = 2$ (that is, $f'(2) > 0$).

In fact, we can be more explicit- We can estimate $f(2.1)$ to be $(0.1)(1.5)$ greater than -3.

(c) Similarly, we expect that $f'(2.1)$ will be LESS THAN 1.5, because f' is decreasing at $x = 2$ (that is, $f''(2) = -0.25$).

In fact, as before, we can estimate $f'(2.1)$ to be $(0.1)(-0.25)$ from 1.5.

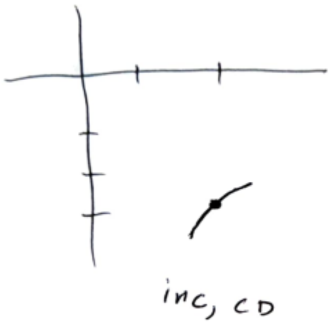
(d) Find the equation of the tangent line to $f(x)$ at $x = 2$.

SOLUTION: The slope is 1.5 and the point that the line goes through is $(2, -3)$, so the line is:

$$y + 3 = 1.5(x - 2)$$

(e) Sketch a graph of $f(x)$ near the point $(2, f(2))$.

SOLUTION:

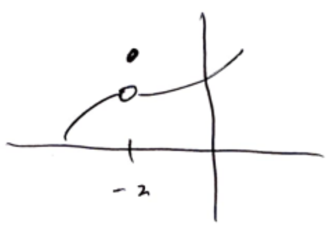


22. For each of the following prompts, give an example of a function that satisfies the stated criteria. A formula or a graph is sufficient for each. If no such example is possible, explain why.

    (a) A function f that is continuous at $a = 2$, but is not differentiable at $a = 2$.
    SOLUTION: A classic example would be the absolute value function, where the vertex has been shifted over to $x = 2$. Therefore, $f(x) = |x - 2|$ would do it.

    (b) A function g that is differentiable at $a = 3$, but does not have a limit at $a = 3$.
    SOLUTION: If a function does not have a limit at $a = 3$, then that function is not continuous at $a = 3$. If a function is not continuous at a point, it cannot be differentiable there either. Therefore, an example where f is differentiable at $x = a$ but doesn't have a limit is impossible.

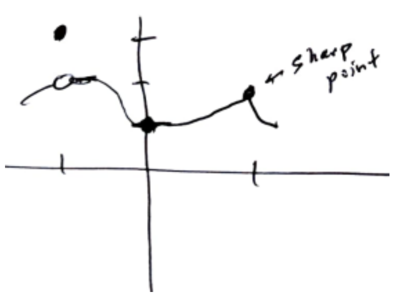
    (c) A function h that has a limit at $a = -2$, is defined at $a = -2$, but is not continuous at $a = -2$.
    SOLUTION: One way to have a limit but not be continuous at $a = -2$ is to set up a function with a hole in the graph at $a = -2$. The graph below is an example.



    (d) A function p that satisfies all of the following:

    i. $p(-1) = 3$, $\lim_{x\to -1} p(x) = 2$
    ii. $p(0) = 1$ and $p'(0) = 0$
    iii. $\lim_{x\to 1} p(x) = p(1)$
    iv. $p'(1)$ does not exist.

    SOLUTION: See the graph below.



23. The graph below is of a function $y = f(x)$, and 5 points are shown. Select the correct signs for each of f, f' and f'' at each point. If the graph is too small, see Exercise 2 of Section 1.6 (AC).

SOLUTION:

	A	B	C	D	E
f	0	$-$	0	$+$	$+$
f'	$-$	0	$+$	0	$-$
f''	$+$	$+$	$+$	$-$	$-$