Sample Final 1 Solutions and Comments

1. Find the domain of $f(x) = \dfrac{\sqrt{x^2-1}}{x-2}$

   SOLUTION: Use a sign chart for the numerator (the denominator is fine everywhere except at $x = 2$).

$x-1$	$-$	$-$	$+$
$x+1$	$-$	$+$	$+$
	$x < -1$	$-1 < x < 1$	$x > 1$

   So the domain is $(-\infty, -1) \cup (1, 2) \cup (2, \infty)$

2. Evaluate the following:

   (a) $\sin(5\pi/6) = 1/2$
   SOLUTION: There was a typo in the original ($5\pi/12$ instead of $5\pi/6$). For this angle, use the unit circle.

   (b) $\tan^{-1}(1/\sqrt{3})$
   SOLUTION: We want θ in the restricted domain $(-\pi/2, \pi/2)$ so that $\tan(\theta) = 1/\sqrt{3}$. Recall that, if (x, y) is a point on the unit circle, then the tangent will be y/x. The angle is $\pi/6$.

   (c) $\log_5(1/\sqrt[3]{25})$
   SOLUTION: Using properties of logs

$$\log_5(1/\sqrt[3]{25}) = \log_5(25^{-1/3}) = \log_5(5^{-2/3}) = -\frac{2}{3}$$

3. True or False (and give a short reason):

   (a) If f is continuous at $x = a$, then f is differentiable at $x = a$.
   False. For example, $f(x) = |x|$ at $x = 0$.

   (b) If $3 \le f(x) \le 5$ for all x, then $6 \le \displaystyle\int_1^3 f(x)\, dx \le 10$.
   True. This is a property of functions ($m \le f(x) \le M$).

   (c) All continuous functions have antiderivatives.
   True. This is the FTC- our first line was: Let f be continuous on $[a, b]$. Then ...

   (d) $\displaystyle\int_{-2}^{1} -x^{-2}\, dx = x^{-1}\Big|_{-2}^{1} = \frac{3}{2}$
   False. Since $1/x^2$ is not continuous on the interval $[-2, 1]$, we cannot use the FTC.

4. Find $f'(2)$ directly from the definition of the derivative, if $f(x) = \frac{1}{x}$ (you may not use l'Hospital's rule).

   SOLUTION:

$$\lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{2-(2+h)}{2(2+h)} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{2(2+h)} = \lim_{h\to 0} \frac{-1}{2(2+h)} = \frac{-1}{4}$$

5. Compute the following limits, where you may not use l'Hospital's rule:

   (a) $\displaystyle\lim_{x\to 2} \frac{x^2 - x - 2}{x - 2}$
   SOLUTION: Factor and cancel

$$\lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2}(x+1) = 3$$

(b) $\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}$
SOLUTION: Multiply by the conjugate:

$$\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})} = \frac{1}{2\sqrt{x}}$$

6. Find the limit, if it exists (you may use any technique from class):

(a) $\lim_{x\to -\infty} \sqrt{\frac{2x^2-1}{x+8x^2}}$
SOLUTION: Divide the numerator and denominator by $-\sqrt{x^2}$:

$$\lim_{x\to -\infty} \sqrt{\frac{(2x^2-1)/x^2}{(x+8x^2)/x^2}} = \lim_{x\to -\infty} \sqrt{\frac{2-\frac{1}{x^2}}{\frac{1}{x}+8}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

(b) $\lim_{x\to 9} \frac{\sqrt{x}}{x-9}$
SOLUTION: If we try to put in $x = 9$, we get "3/0", which means that x will be approaching $\pm\infty$. In this case, we can say that the limit does not exist.

7. Differentiate:

(a) $\frac{2x}{\sqrt{x^2+1}}$

$$\frac{2(x^2+1)-2x(2x)}{(x^2+1)^2}$$

(b) $y = \sin^3(x^2+1)$

$$\frac{dy}{dx} = 3\sin^2(x^2+1)\cos(x^2+1)(2x)$$

(c) $y = x^x$
Logarithmic differentiation:

$$\ln(y) = \ln(x^2) = x\ln(x) \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = 1\ln(x) + x\frac{1}{x} \quad \Rightarrow \quad \frac{dy}{dx} = (x^x)(\ln(x)+1)$$

8. Find the equation of the tangent line to $\sqrt{x+y} = \frac{1}{2}xy$ at the point $(2, 2)$.
Implicit differentiation for the slope

$$\frac{1}{2}(x+y)^{-1/2}(1+y') = \frac{1}{2}(y+xy') \quad \Rightarrow \quad \frac{1}{4}(1+y') = \frac{1}{2}(2+2y') \quad \Rightarrow \quad y' = -1$$

Given that slope and the point $(2, 2)$, the equation of the line is

$$y-2 = -1(x-2) \quad \Rightarrow \quad y = -x+4$$

9. Evaluate the Riemann sum by first writing it as an appropriate definite integral: $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3}{n}\sqrt{1+\frac{3i}{n}}$.

SOLUTION: The length of the interval is 3. A natural choice for a is $a = 1$, so that

$$a + i\frac{b-a}{n} = 1 + \frac{3i}{n}$$

which is the expression under the square root (so that $f(x) = \sqrt{x}$). This gives us the definite integral

$$\int_1^4 \sqrt{x}\, dx = \frac{2}{3}x^{3/2}\Big|_0^3 = \frac{2}{3}(4^{3/2}-1) = \frac{14}{3}$$

10. Differentiate: $F(x) = \int_{\sqrt{x}}^{x^2} \frac{t}{1+t}\, dt$ SOLUTION:

$$\frac{x^2}{1+x^2}(2x) - \frac{\sqrt{x}}{1+\sqrt{x}} \cdot \frac{1}{2}x^{-1/2}$$

11. Write the definite integral as an appropriate Riemann sum (using right endpoints): $\int_0^3 1 + 3x^2\, dx$

SOLUTION:

$$\lim_{n\to\infty} frac3n \sum_{i=1}^{n} 1 + 3\left(\frac{3i}{n}\right)^2$$

12. Evaluate, or find the general indefinite integral.

(a) $\int \sqrt{x^3} + \frac{1}{x^2+1}\, dx = \frac{2}{5}x^{5/2} + \tan^{-1}(x) + C$

(b) $\int_{-1}^{1} t(1-t)\, dt = \int_{-1}^{1} t - t^2\, dt = \frac{1}{2}t^2 - \frac{1}{3}t^3\Big|_{-1}^{1} = \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{2} + \frac{1}{3}\right) = -\frac{2}{3}$

(c) $\int_0^1 5x - 5^x\, dx = \frac{5}{2}x^2 - \frac{5^x}{\ln(5)}\Big|_0^1 = \left(\frac{5}{2} - \frac{5}{\ln(5)}\right) - \left(0 - \frac{1}{\ln(5)}\right) = \frac{5}{2} - \frac{4}{\ln(5)}$

13. Evaluate:

(a) $\int_0^1 \frac{d}{dx}\left(e^{\tan^{-1}(x)}\right)\, dx = e^{\tan^{-1}(1)} - e^{\tan^{-1}(0)} = e^{\pi/4} - 1$

(b) $\frac{d}{dx}\int_0^1 e^{\tan^{-1}(x)}\, dx = 0$ (the definite integral is a constant).

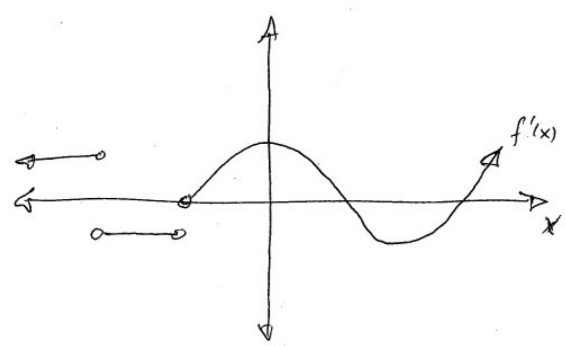
(c) $\frac{d}{dx}\int_0^x e^{\tan^{-1}(t)}\, dt = e^{\tan^{-1}(x)}$

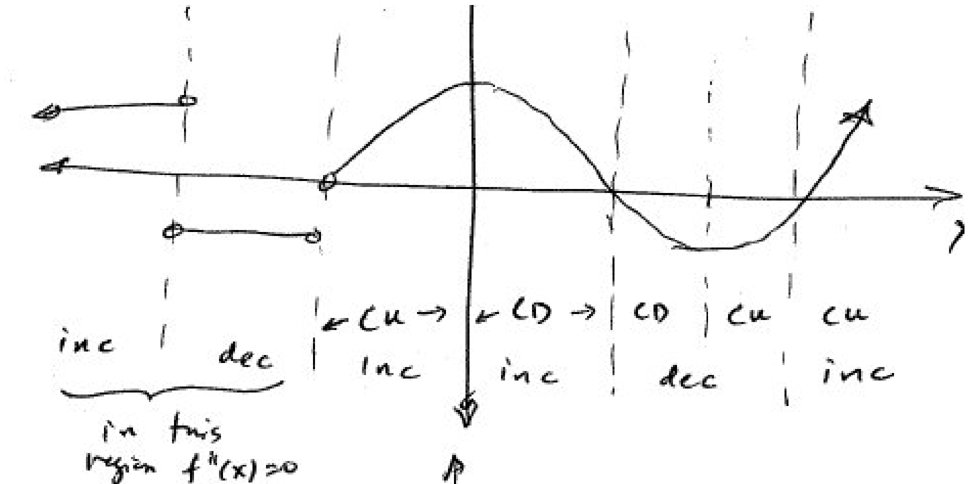
14. Given the graph of the derivative, $f'(x)$, below, answer the following questions:

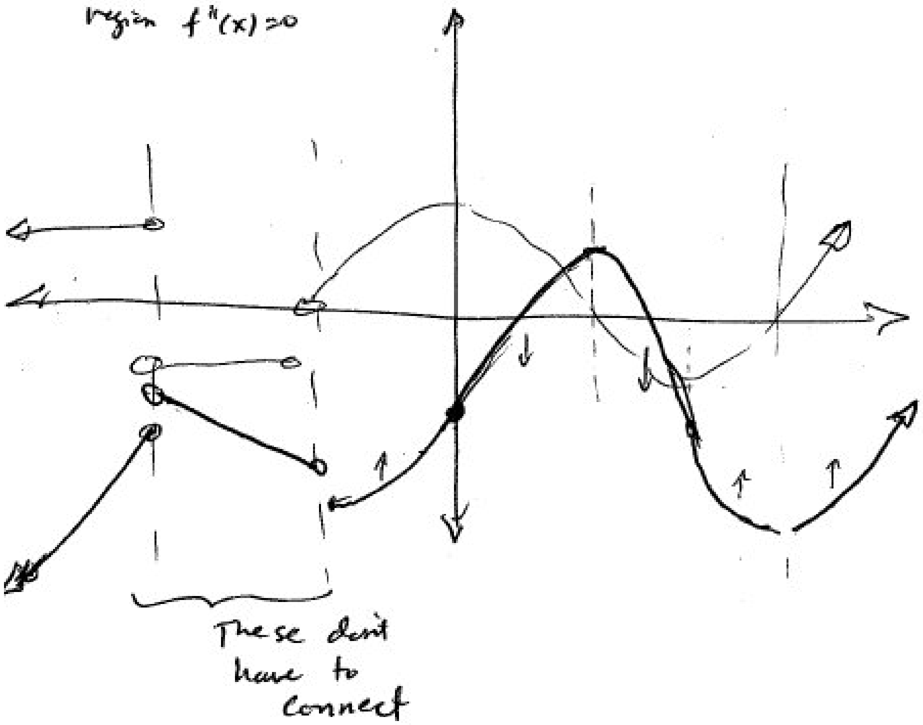
(a) Find all intervals on which f is increasing.

(b) Find all intervals on which f is concave up.

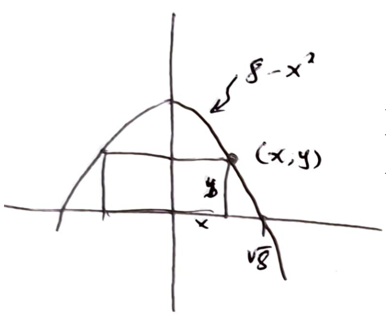
(c) Sketch a possible graph of f if we require that $f(0) = -1$.







15. A rectangle is to be inscribed between the $x-$axis and the upper part of the graph of $y = 8 - x^2$ (symmetric about the $y-$axis). For example, one such rectangle might have vertices: $(1, 0), (1, 7), (-1, 7), (-1, 0)$ which would have an area of 14. Find the dimensions of the rectangle that will give the largest area.



SOLUTION: Let (x, y) be a point on the parabola that is also a corner of the rectangle. Then the rectangle has dimensions $2x \times y$.
We want to maximize the area $A = 2xy$ such that $y = 8 - x^2$. Therefore, we can substitute y in the area expression so that the area is now:

$$A(x) = 2x(8 - x^2) = 16x - 2x^3, \quad 0 \le x \le \sqrt{8}$$

We would normally construct a table of values using endpoints and critical points, however, the area at $x = 0$ and and $x = \sqrt{8}$ are both zero. Therefore, if we get any positive area for the critical point, it will be the max. The critical points are

$$A'(x) = 16 - 6x^2 = 0 \quad \Rightarrow \quad x^2 = \frac{8}{3} \quad \Rightarrow \quad x = \sqrt{\frac{8}{3}}$$

With this, $y = 8 - x^2 = 8 - \frac{8}{3} = \frac{16}{3}$, so the dimensions of the box giving the maximum area are

$$2 \cdot \sqrt{\frac{8}{3}} \times \frac{16}{3}$$

16. Find all values of c and d so that f is continuous at all real numbers:

$$f(x) = \begin{cases} 2x^2 - 1 & \text{if } x < 0 \\ cx + d & \text{if } 0 \le x \le 1 \\ \sqrt{x+3} & \text{if } x > 1 \end{cases}$$

Be sure it is clear from your work that you understand the definition of continuity.

SOLUTION: We only need to check the definition of continuity at each break point- In this case, at $x = 0$ and $x = 1$. There are three things to check: (i) Does $f(a)$ exist? (ii) Does the limit exist? (we'll need to take the limit from the left and right separately), and (iii) Parts (i) and (ii) should be the same number. Here we go:

- At $x = 0$:
  - $f(0) = d$
  - $\lim_{x \to 0^-} 2x^2 - 1 = -1$ and $\lim_{x \to 0^+} cx + d = d$
    By the two computations above, the limit exists if $d = -1$ (and therefore the limit is -1.
  - If $d = -1$ then (i)=(ii), and the function is continuous at $x = 0$.
- Check $x = 1$:
  - $f(1) = c + d = c - 1$
  - $\lim_{x \to 1^-} cx - 1 = c - 1$ and $\lim_{x \to 0^+} \sqrt{x+3} = \sqrt{4} = 2$
    For the limit to exist, $c - 1 = 2$, or $c = 3$. In that case, the limit and the value of the function are both 2.
  - Parts (i) and (ii) are the same.

Conclusion: For $c = 3$ and $d = -1$, the function is continuous at all x.

17. A perfectly round balloon is being inflated at a rate of 2000 cubic cm per second. How fast is the radius of the balloon increasing when the radius is 10 cm? (Volume of a sphere is $V = \frac{4}{3}\pi r^3$).

SOLUTION: Both volume and radius are functions of time, so that

$$V(t) = \frac{4}{3}\pi(r(t))^3 \quad \Rightarrow \quad \frac{dV}{dt} = 4\pi(r(t))^2 \frac{dr}{dt}$$

From what we're given, $dV/dt = 2000$ and we want to find dr/dt when $r = 10$:

$$2000 = 4\pi(10)^2 \frac{dr}{dt} \quad \Rightarrow \quad \frac{dr}{dt} = \frac{5}{\pi} \text{ cm/sec}$$

18. Suppose that over 45 months, the amount of radioactive element in a sample goes from 4 grams to $\frac{1}{2}$ gram. (i) Find a formula for the amount present at any time t. (ii) What is the half-life of the element? (Leave answers in exact form)

SOLUTION: Our model equation is either $f(t) = Ce^{kt}$ or $f(t) = C \cdot 2^{kt}$. We'll use the the first function below.

The value of C is the initial amount of material ($C = 4$). We can use the data given to determine the value of k:

$$\frac{1}{2} = 4e^{45\,k} \quad \Rightarrow \quad \ln(1/8) = 45 \cdot k \quad \Rightarrow \quad k = -\ln(8)/45$$

(Note: You can keep k as $ln(1/8)/45$ if you want to). With this, the answer to part (i) is:

$$f(t) = 4e^{-\ln(8)\,t/45}$$

For the half-life, solve for t:

$$2 = 4e^{-\ln(8)\,t/45} \quad \Rightarrow \quad -\ln(2) = \frac{-\ln(8)\,t}{45} \quad \Rightarrow \quad t = \frac{45\ln(2)}{\ln(8)}$$

You can leave your answer in this form, but you might try (for practice) to show that $t = 15$.