Sample Final Exam 2 Solutions

1. Short Answer:

   (a) True or False? $\dfrac{x^2-1}{x-1} = x+1$
   SOLUTION: False, unless we restrict both expressions to all x except $x=1$- That is, the expression on the left is not defined at $x=1$, but the expression on the right is defined at $x=1$.

   (b) If $f'(2)$ exists, then $\lim_{x\to 2} f(x) = f(2)$
   SOLUTION: (Typo: Should have started with "True or False") True- This says that if the derivative of f exists at $x=2$, then f is continuous at $x=2$ (which is a theorem we know).

   (c) If $f(x) = (2-3x)^{-1/2}$, find $f(0), f'(0)$ and $f''(0)$.
   SOLUTION: Just do the computation. $f(0) = 2^{-1/2} = \frac{1}{\sqrt{2}}$, and

   $$f'(x) = -\frac{1}{2}(2-3x)^{-3/2}(-3) = \frac{3}{2(2-3x)^{3/2}} \quad \Rightarrow \quad f'(0) = \frac{3}{2^{5/2}}$$

   $$f''(x) = -\frac{9}{4}(2-3x)^{-5/2}(-3) \quad \Rightarrow \quad f''(0) = \frac{27}{4\cdot 2^{5/2}} = \frac{27}{2^{9/2}}$$

   (d) Show that the $x^4+4x+c=0$ has at most one solution in the interval $[-1,1]$.
   SOLUTION: Let $f(x) = x^4+4x+c$. Then $f'(x) = 4x^3+4$, and we note that $f'(x)=0$ only at $x=-1$. We know, by the Mean Value Theorem (or more specifically, Rolle's Theorem) that if $f(x)=0$ twice or more in $(-1,1]$, then $f'(x)$ would have to be zero at least once in this interval (but it is not). Therefore, f can be zero at most once in $(-1,1]$. We can also note that if $f(x)=0$ at $x=-1$, then again $f(x)$ cannot be zero again in the interval for the same reason (the derivative would have to be zero somewhere in $(-1,1]$).

2. Differentiate:

   (a) $y = xe^{3x}$
   SOLUTION: Product rule
   $$y' = e^{3x} + x\cdot 3e^{3x}$$

   (b) $y = 4^{1/x} + \sin^{-1}(3x+1)$
   SOLUTION:
   $$\frac{dy}{dx} = 4^{1/x}\ln(4)\cdot\frac{-1}{x^2} + \frac{1}{\sqrt{1-(3x+1)^2}}\cdot 3$$

   (c) $y = \frac{x^2-1}{\sqrt{x}}$
   SOLUTION: You could use the quotient rule, but with a little algebra we don't need it:
   $$y = \frac{x^2}{x^{1/2}} - \frac{1}{x^{1/2}} = x^{3/2} - x^{-1/2}$$
   Now we can differentiate:
   $$y' = \frac{3}{2}x^{1/2} + \frac{1}{2}x^{-3/2}$$

3. Find the equation to the tangent line for $\sqrt{x}+\sqrt{y}+xy=3$ at the point $(1,1)$:
   SOLUTION: Use implicit differentiation:
   $$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' + y + xy' = 0 \quad\Rightarrow\quad \frac{1}{2}+\frac{1}{2}y'+1+y'=0 \quad\Rightarrow\quad y'=-1$$
   Therefore the equation of the line is
   $$y-1 = -1(x-1) \quad \text{or } y = -x+2$$

4. Find $f'(4)$ directly from the definition of the derivative (using limits and without l'Hospital's rule): $f(x) = \sqrt{x}$

SOLUTION:

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} = \lim_{h\to 0} \frac{\sqrt{4+h}-2}{h} \cdot \frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} =$$

$$\lim_{h\to 0} \frac{4+h-4}{h(\sqrt{4+h}+2)} = \lim_{h\to 0} \frac{1}{\sqrt{4+h}+2} = \frac{1}{4}$$

5. Find the limit if it exists. You may use any method (except for a numerical table).

   (a) $\lim_{x\to\infty} \sqrt{9x^2+x} - 3x$

   SOLUTION: Multiply by the conjugate and we'll introduce a fraction:

$$\lim_{x\to\infty} (\sqrt{9x^2+x}-3x)\frac{\sqrt{9x^2+x}+3x}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \frac{9x^2+x-9x^2}{\sqrt{9x^2+x}+3x} = \lim_{x\to\infty} \frac{x/x}{(\sqrt{9x^2+x}+3x)/x}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{9+\frac{1}{x}}+3} = \frac{1}{6}$$

   (b) $\lim_{x\to\pi^-} \frac{\sin(x)}{1-\cos(x)}$

   SOLUTION: Try evaluating first, and we get $0/(1 - -1) = 0/2 = 0$.

   (c) $\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)}$

   SOLUTION: Use l'Hospital's Rule:

$$\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)} = \lim_{x\to 0} \frac{1}{\frac{1}{1+(4x)^2}\cdot 4} = \lim_{x\to 0} \frac{1+16x^2}{4} = \frac{1}{4}$$

6. Differentiate: $F(x) = \int_{2x}^{x^2} e^{t^2}\, dt$

   SOLUTION: Given $F(x) = \int_{h_1(x)}^{h_2(x)} f(t)\, dt$, then

$$F'(x) = f(h_1(x))h_1'(x) - f(h_2(x))h_2'(x)$$

   In this case,

$$F'(x) = e^{x^4}(2x) - e^{4x^2}(2)$$

7. Write the definite integral as an appropriate Riemann sum: $\int_2^5 x^2+1\, dx$

   SOLUTION: $f(x) = x^2+1$, $b = 5$ and $a = 2$. Therefore,

$$\int_2^5 x^2+1\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(2+\frac{3i}{n}\right)^2+1\right)\left(\frac{3}{n}\right)$$

8. Evaluate the integral, if it exists

   (a) $\int_1^9 \frac{\sqrt{u}-2u^2}{u}\, du = \int_1^9 u^{-1/2} - 2u\, du = 2u^{1/2} - u^2\Big|_1^9 = (6-81)-(2-1) = -76$

(b) $\displaystyle\int 3^x + \frac{1}{x} + \sec^2(x)\,dx = \frac{3^x}{\ln(3)} + \ln|x| + \tan(x) + C$

(c) $\displaystyle\int_{\pi/4}^{\pi/4} \frac{t^4\tan(t)}{2+\cos(t)}\,dt = 0$

(d) $\displaystyle\int_0^3 |x^2-4|\,dx$

For the last problem, we need to get rid of the absolute value, and re-write as:

$$|x^2-4| = \begin{cases} x^2-4 & \text{if } x<-2 \text{ or } x>2 \\ -x^2+4 & \text{if } -2\le x\le 2 \end{cases}$$

Therefore,

$$\int_0^3 |x^2-4|\,dx = \int_0^2 -x^2+4\,dx + \int_2^3 x^2+4\,dx = -\frac{1}{3}x^3+4x\Big|_0^2 + \frac{1}{3}x^2-4x\Big|_2^3 = \frac{23}{3}$$

Grading note: Don't let the arithmetic slow you down! If you've gotten everything except the final arithmetic computation, come back to that if you have time at the end.

9. A water tank in the shape of an inverted cone with a circular base has a base radius of 2 meters and a height of 4 meters. If water is being pumped into the tank at a rate of 2 cubic meters per minute, find the rate at which the water level is rising when the water is 3 meters deep. ($V = \frac{1}{3}\pi r^2 h$)

SOLUTION: If h is the height of the water, and r is the radius, the volume of water is a different than what is given for the whole cone. The volume of water is the volume of the whole cone minus the "empty" cone. That is, if h is the height of the water, then the volume is:

$$V = \frac{1}{3}\pi 2^2\cdot 4 - \frac{1}{3}\pi r^2(4-h)$$

From similar triangles, we get $r = \frac{1}{2}(4-h)$, so our formula becomes:

$$V = \frac{16}{3}\pi - \frac{\pi}{12}(4-h)^3$$

Now, treat V, h as functions of time:

$$\frac{dV}{dt} = -\frac{\pi}{12}3(4-h)^2(-1) = \frac{\pi}{4}(4-h)^2\frac{dh}{dt}$$

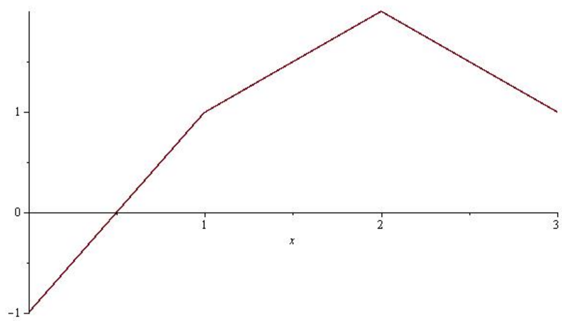
With $dV/dt = 2$ and $h=3$, we get: $dh/dt = 8/\pi$.

10. Explain why the following is true (if it is): The function $f(x) = \sqrt{1+2x}$ can be well approximated by $(1+x)/3$ if x is approximately 4.

This is the equation of the tangent line to f at $x=4$:

$$f(4) = \sqrt{1+8} = 3 \qquad f'(4) = \frac{1}{\sqrt{1+8}} = \frac{1}{3} \quad\Rightarrow\quad L(x) = 3 + \frac{1}{3}(x-4) = \frac{x+5}{3}$$

11. For the solution, start at $(0,-1)$, then draw a line with slope 2 for one unit over. This puts us at the point $(1,1)$. From there, draw a line with slope 1 for one unit- That gets us to $(2,2)$. Finally, draw a line with slope -1 for one more unit, and that puts at the final point of $(3,1)$.



12. Find m and b so that f is continuous and differentiable:

$$f(x) = \begin{cases} x^2 \text{ if } x \le 2 \\ mx + b \text{ if } x > 2 \end{cases}$$

SOLUTION: We note that the derivative is:

$$f'(x) = \begin{cases} 2x \text{ if } x < 2 \\ m \text{ if } x > 2 \end{cases}$$

Therefore, if we make $m = 4$, the function will be differentiable at $x = 2$. However, in order to be differentiable, the function needed to be continuous as well- Now that $m = 4$, we check to see if f is continuous at $x = 2$ by going through the definition of continuity:

- Does $f(2)$ exist? Yes. $f(2) = 2^2 = 4$.
- Does the limit exist at $x = 2$?
  - From the left: $\lim_{x\to 2^-} f(x) = 2^2 = 4$
  - From the right: $\lim_{x\to 2^+} f(x) = 4(2) + b = 8 + b$

  Therefore, the limit exists (and is $f(2)$) if $8 + b = 4$, or $b = -4$.

The function f should be:

$$f(x) = \begin{cases} x^2 \text{ if } x \le 2 \\ 4x - 4 \text{ if } x > 2 \end{cases}$$

You may note that $4x - 4$ is the tangent line to x^2 at $x = 2$ as well.

13. Boat A is traveling north at a constant speed of 10 kilometers per hour. At noon, boat B is located 10 km east of boat A and boat B is traveling east at a constant speed of 10 kilometers per hour. How fast is the distance between the boats increasing at 3:00 PM?

SOLUTION: Consider the position of Boat A at noon. Let x be the distance from Boat A to this point, and let y be the distance from that point to Boat B. If z is the distance between the two boats, then

$$z^2 = x^2 + y^2 \quad \Rightarrow \quad 2zz' = 2xx' + 2yy' \quad \Rightarrow \quad z' = \frac{xx' + yy'}{z} = \frac{300 + 400}{50} = 14 \text{ km per hour}$$

14. Suppose that over a period of 70 years, the population of a country goes from 20 million to 80 million. If the growth is exponential, find the doubling time of the population. Find a formula for the population at any time t.

SOLUITON: In the first sample exam, we used $f(t) = Ce^{kt}$, so in this sample, we'll use the alternative $f(t) = C \cdot 2^{kt}$ (either one is fine).

Therefore,

$$80 = 20 \cdot 2^{70\,k} \quad \Rightarrow \quad 4 = 2^{70k} \quad \Rightarrow \quad \log_2(2^2) = 70k \quad \Rightarrow \quad k = \frac{2}{70} = \frac{1}{35}$$

Therefore, our function is $f(t) = 20 \cdot 2^{t/35}$. For the doubling time,

$$40 = 20 \cdot 2^{t/35} \quad \Rightarrow \quad 2^1 = 2^{t/35}$$

therefore, $t = 35$ years for the doubling time.

15. What is the minimum possible surface area of a rectangular box with square base and a volume of 8 cubic feet?

SOLUTION: If x is the width (and length), and y is the height of the box, then we want to minimize the surface area given a fixed volume of 8:

$$\min\ A = 2x^2 + 4xy \quad \text{ such that } 8 = x^2 y$$

Now we can substitute for y: $y = 8/x^2$ to get A in terms of x alone

$$A(x) = 2x^2 + 4x\frac{8}{x^2} = 2x^2 + \frac{32}{x}$$

Now compute the critical point: $A'(x) = 4x - 32/x^2 = 0$ which gives $x = y = 2$, so it is a cube with surface area 24.

16. If $f(x) = 1/x$ on the interval $[1, 2]$, find the point(s) c that are guaranteed by the Mean Value Theorem.

SOLUTION: We want to solve the equation

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Substituting our values:

$$-\frac{1}{c^2} = \frac{1/2 - 1}{2 - 1} = -\frac{1}{2}$$

so that $c = \sqrt{2}$, which is in our interval.