Sample Exam 3 Solutions

1. Evaluate:

   (a) $\sec(-9\pi/4) = \sqrt{2}$

   (b) $\sin^{-1}(\sqrt{3}/2) = \pi/3$

   (c) $\log_7(\sqrt{7}/49^3)$
   SOLUTION: In the parenthesis,

   $$\frac{\sqrt{7}}{49^3} = \frac{7^{1/2}}{(7^2)^3} = 7^{1/2-6} = 7^{-11/2}$$

   so the answer is $-11/2$.

2. True or False, and give a short reason:

   (a) $\frac{d}{dx}(10^x) = x10^{x-1}$
   SOLUTION: FALSE. The power rule for derivatives is only valid when the exponent is a constant (and the base is a variable). In this case, the derivative ought to be: $10^x\ln(10)$.

   (b) $\displaystyle\lim_{x\to 1}\frac{x^2+6x-7}{x^2+5x-6} = \frac{\lim_{x\to 1} x^2+6x-7}{\lim_{x\to 1} x^2+5x-6}$
   FALSE: It would be true if the denominator was not zero. Additionally, we see that the limit could be taken by either factoring out $(x-1)$ from the numerator and denominator, or by using l'Hospital's rule (which gives 8/7).

   (c) If $f(x) = x^2$, then the equation of the normal line at $x = 3$ is: $y - 9 = \frac{-1}{2x}(x-3)$
   FALSE: It would be true if we EVALUATED the negative reciprocal of the derivative at $x = 3$. That is, the following is the equation of the normal line: $y - 9 = \frac{-1}{6}(x-3)$

3. Differentiate:

   (a) $y = 3^{x^2-1} + (x^2-3x+1)^5$
   SOLUTION:
   $$y' = 3^{x^2-1}\ln(3)(2x) + 5(3x^2-3x+1)^4(2x-3)$$

   (b) $y = \dfrac{1-2x}{\sqrt[3]{x^5}}$
   SOLUTION: Simplify first: $y = x^{-5/3} - 2x^{-2/3}$, so that

   $$y' = -\frac{5}{3}x^{-8/3} + \frac{4}{3}x^{-5/3}$$

   (c) $y = (x^2-1)^{\sin(x)}$
   SOLUTION: Logarithmic differentiation:

   $$y = (x^2-1)^{\sin(x)} \quad\Rightarrow\quad \ln(y) = \sin(x)\ln(x^2-1)$$

so that

$$\frac{y'}{y} = \cos(x)\ln(x^2-1) + \sin(x)\frac{2x}{x^2-1}$$

Simplify:

$$y' = (x^2-1)^{\sin(x)}\left(\cos(x)\ln(x^2-1) + \frac{2x\sin(x)}{x^2-1}\right)$$

4. Find all vertical and horizontal asymptotes of $f(x) = \dfrac{2x^2-2}{x^2-x-2}$

   SOLUTION: For the vertical asymptotes, it is clearer if we factor first:

$$\frac{2x^2-2}{x^2-x-2} = \frac{2(x-1)(x+1)}{(x+1)(x-2)} = \frac{2(x-1)}{(x-2)} \quad \text{for } x \neq -1$$

   Therefore, there is a hole in the graph at $x = -1$, and a vertical asymptote at $x = 2$. To find the horizontal asymptote, take the limit out to $\pm\infty$, which is easy to compute via l'Hospital's rule to be 2 (therefore, the horizontal asymptote is $y = 2$).

5. Derive the formula for the derivative of $y = \sec^{-1}(x)$:

   SOLUTION: First re-write this as $\sec(y) = x$. This defines the three lengths of a right triangle where one of the acute angles is y, the length of the hypotenuse is x, and the length of the side adjacent is 1. Therefore, the length of the third side (the opposite) is $\sqrt{x^2-1}$. Now differentiate the expression implicitly, and back-substitute using the triangle:

$$\sec(y)\tan(y)y' = 1 \quad \Rightarrow \quad y' = \frac{1}{\sec(y)\tan(y)} = \frac{1}{x\sqrt{x^2-1}}$$

6. Find $f'(1)$ using the definition of the derivative (using limits and you may not use l'Hospital's rule), if $f(x) = \dfrac{x}{x+1}$

   NOTE: Often it is easier to compute the derivative at a specific number if you evaluate at the number. You'll see it below.

   Use the definition, then get a common denominator (and simplify):

$$f'(1) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{\frac{1+h}{2+h}-\frac{1}{2}}{h} = \lim_{h\to 0}\frac{1}{h}\left(\frac{2(1+h)-(2+h)}{2(2+h)}\right) = \frac{1}{4}$$

7. Find the limit, if it exists (you may use any method from class):

   (a) $\displaystyle\lim_{x\to 3}\frac{\sqrt{x+6}-x}{x^3-3x^2}$

   SOLUTION: Multiply by the conjugate and factor the denominator:

$$\lim_{x\to 3}\frac{\sqrt{x+6}-x}{x^3-3x^2}\cdot\frac{\sqrt{x+6}+x}{\sqrt{x+6}+x} = \lim_{x\to 3}\frac{-x^2+x+6}{x^2(x-3)(\sqrt{x+6}+x)} =$$

$$\lim_{x\to 3}\frac{-(x-3)(x+2)}{x^2(x-3)(\sqrt{x+6}+x)} = \lim_{x\to 3}\frac{-(x+2)}{x^2(\sqrt{x+6}+x)} = -\frac{5}{54}$$

(b) $\lim_{x\to-\infty} \dfrac{\sqrt{x^2-9}}{2x-6}$

SOLUTION: Divide by x, then remember to substitute $x = -\sqrt{x^2}$ since $x < 0$:

$$\lim_{x\to-\infty} \frac{\sqrt{x^2-9}/x}{(2x-6)/x} = \lim_{x\to-\infty} \frac{-\sqrt{\frac{x^2-9}{x^2}}}{2-\frac{6}{x}} = \lim_{x\to-\infty} \frac{-\sqrt{1-\frac{9}{x^2}}}{2-\frac{6}{x}} = -\frac{1}{2}$$

(c) $\lim_{x\to0^+} \left(\dfrac{1}{x} - \dfrac{1}{e^x-1}\right)$

SOLUTION: Combine into a single fraction first, then see if l'Hospital's rule will work

$$\lim_{x\to0^+} \frac{\mathrm{e}^x-1-x}{x(\mathrm{e}^x-1)} = \lim_{x\to0^+} \frac{\mathrm{e}^x-1}{(\mathrm{e}^x-1)+x\mathrm{e}^x} = \lim_{x\to0^+} \frac{\mathrm{e}^x}{\mathrm{e}^x+\mathrm{e}^x+x\mathrm{e}^x} = \frac{1}{2}$$

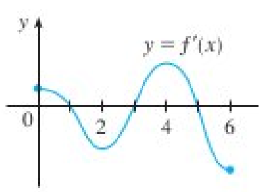
8. Find f if $f''(x) = 6x + \sin(x)$ if $f'(0) = 0$ and $f(0) = 3$.

SOLUTION: Antidifferentiate twice, using the information provided to solve for the arbitrary constants:

$$f'(x) = 3x^2 - \cos(x) + C_1 \quad\Rightarrow\quad f'(x) = 3x^2 - \cos(x) + 1$$

$$f(x) = x^3 - \sin(x) + x + C_2 \quad\Rightarrow\quad f(x) = x^3 - \sin(x) + x + 3$$

9. The following is the graph of $f'(x)$:



(a) On what intervals is f increasing or decreasing?

SOLUTION: The function f is increasing where the graph of $f'(x)$ is above the $x-$axis: $(0,1),(3,5)$.

The function f is decreasing where the graph of $f'(x)$ is below the $x-$axis: $(1,3),(5,6)$

(b) On what intervals is f concave up or concave down?

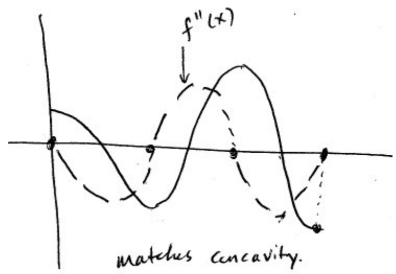
SOLUTION: The function f is concave up where the graph of $f'(x)$ is increasing: $(2,4)$.

The function f is concave down where the graph of $f'(x)$ is decreasing: $(0,2)$ and $(4,5)$.

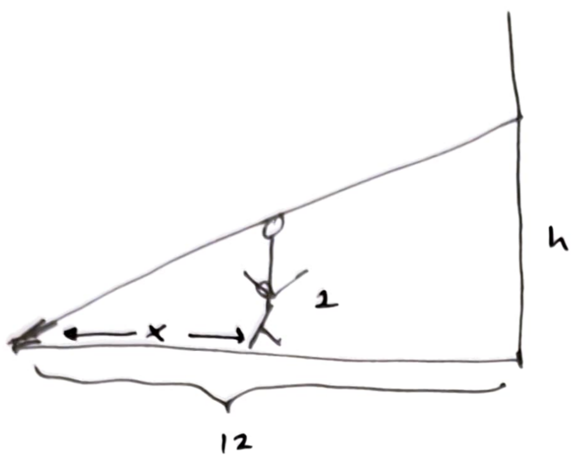
(c) At what points does f have a local maximum?

SOLUTION: f has a local max that occurs at both $x = 1$ and $x = 5$ (the derivative goes from positive to negative- That is the first derivative test).

(d) Sketch a graph of f''.



10. A spotlight on the ground shines one a wall 12 meters away. If a man 2 meters tall walks from the spotlight to the wall at a speed of 1.6 meters per second, how fast is the length of the shadow on the building decreasing when he is 4 meters from the building?



SOLUTION: First, make a sketch. The variable here will be x, the distance the man is from the spotlight. Note that when the man is 4 meters from the building, he is 8 meters from the spotlight. Further, a speed of 1.6 will be dx/dt.

Now, using the sketch as a guide, we can use similar triangles to get an expression in h (the question asks us to find dh/dt).

$$\frac{h}{2} = \frac{12}{x} \quad \Rightarrow \quad h = \frac{24}{x} \quad \Rightarrow \quad \frac{dh}{dt} = -\frac{24}{x^2}\frac{dx}{dt} = -\frac{24}{8^2}\frac{8}{5} = -\frac{3}{5}$$

11. It has been estimated that since the second half of the 19th century, the population of the United States doubles approximately every 56 years. If the current population is approximately 311 million, when will the population reach half a billion?

SOLUTION: The first piece of information can be used to find the rate of growth

$$2P = Pe^{56\,r} \quad \Rightarrow \quad r = \frac{\ln(56)}{2} \approx 2.0127$$

Now, how long to reach 500 million?

$$500 = 311e^{rt} \quad \Rightarrow \quad t = \frac{1}{r}\ln\left(\frac{500}{311}\right) = \frac{56\ln(500/311)}{\ln(2)} \approx 38.36 \text{ years}$$

(On an exam or quiz, you can leave your answer in exact form)

Side Remark: Some like to use the model $y(t) = P_0 2^{kt}$ instead if $y(t) = P_0 \mathrm{e}^{rt}$. Starting from the base 2 model, here is how you relate r and k:

Since, for any number $A > 0$, we can express it as: $A = \mathrm{e}^{\ln(A)}$, then

$$2^{kt} = \left(\mathrm{e}^{\ln(2)}\right)^{kt} = \mathrm{e}^{\ln(2)\,kt}$$

so $r = k\ln(2)$, and the two functions are the same.

12. If $F(x) = \int_0^x 6 - 3t\,dt$, find where F is increasing/decreasing.

    SOLUTION: Since $F'(x) = 6 - 3x$, F is increasing when $6 - 3x > 0$, or $x < 2$. The function F would be decreasing when $x > 2$.

13. Evaluate:

    (a) $\displaystyle\int_0^1 (x+2)(x+1)\,dx = \int_0^1 x^2 + 3x + 2\,dx = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 2x\Big|_0^1 = 4 - 0 = 4$

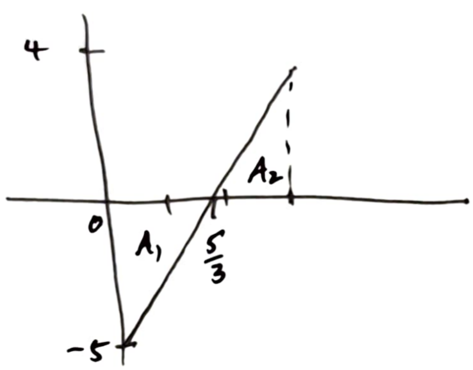
    (b) $\displaystyle\int_1^{18} \sqrt{\frac{3}{z}}\,dz = \int_1^{18} \sqrt{3}z^{-1/2}\,dz = 2\sqrt{3}z^{1/2}\Big|_1^{18} = 2\sqrt{3}(\sqrt{18} - 1)$

    (c) $\int_0^1 x^e + e^x\,dx = \frac{1}{e+1}x^{e+1} + e^x\Big|_0^1 = \left(\frac{1}{e+1} + e\right) - (0+1)$

14. The velocity function for a particle moving along a line is $v(t) = 3t - 5$. (i) Find the displacement on the interval $[0, 3]$, then (ii) find the distance traveled on $[0, 3]$.

    SOLUTION:

    For displacement, $\displaystyle\int_0^3 3t - 5\,dt = \frac{3}{2}t^2 - 5t\Big|_0^3 = -\frac{3}{2}$.



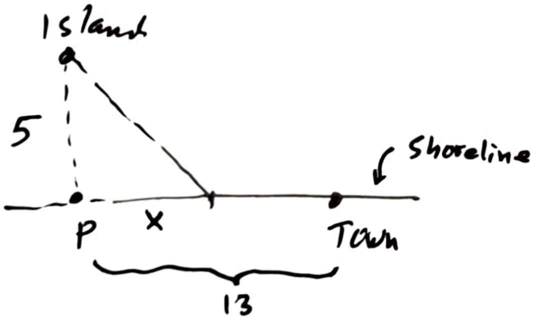
    For distance, note that the curve is under the $x-$axis for $0 \le t \le 5/3$, so we need to add that area in rather than subtract it. It's easiest to use geometry:

$$A_1 = \frac{1}{2}\cdot\frac{5}{3}\cdot 5 = \frac{25}{6} \qquad A_2 = \frac{1}{2}\cdot\frac{4}{3}\cdot 4$$

    Therefore, the distance traveled is $(25 + 16)/6 = 41/6$.

15. The shortest distance from an island to a straight shoreline is 5 km. Let P be the point on the shoreline corresponding to this shortest distance. From point P, there is a town 13 kilometers away along the beach. We can travel at 5 k per hour in the water, and 10 k per hour on the beach. Construct a function that can be used to find the point on the

beach that will minimize our travel time from the island to the town. You do not have to find the minimum, just construct the function.



SOLUTION: See the sketch, and recall that $d = rt$, or $t = d/r$. Therefore, the time to travel to town is the sum of the time on the water plus the time on the shore:

$$T(x) = \frac{\sqrt{5^2 + x^2}}{5} + \frac{13 - x}{10}, \quad 0 \le 0 \le 13$$

As a note, to find the minimum (and maximum), we would construct a table and include the endpoints and critical points, then find the x giving the minimum time.