### The Limit Laws

Suppose that $c$ is a constant and the limits each exist separately:

$$\lim_{x\to a} f(x) = L \qquad \lim_{x\to a} g(x) = H$$

Then:

1., 2. $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L \pm H$

3. $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x) = cL$

4. $\lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = LH$

5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{\lim f(x)}{\lim g(x)} = \dfrac{L}{H}$ for $H \neq 0$. (If $H = 0$, then more work is needed).

6.-11. Some specific functions:

$$\lim_{x\to a} c = c \qquad \lim_{x\to a} x = a \qquad \lim_{x\to a} x^n = a^n \qquad \lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$$

## Theorems to help us compute limits

1. Direct substitution property: If $f$ is a polynomial or a rational function, and $a$ is in the domain of $f$, then the limit is found simply by substitution:

$$\lim_{x\to a} f(x) = f(a)$$

2. If $f(x) \leq g(x)$ when $x$ is near $a$ (except possibly at $x = a$), and the limits of $f, g$ each exist as $x$ approaches $a$, then:

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

3. (The Squeeze Theorem) Let $g(x) \leq f(x) \leq h(x)$ for all $x \in [a, b]$, and let $c \in [a, b]$. If the limit of $g, h$ are both the same number $L$ as $x \to c$, then so is the limit of $f(x)$.

## Algebra and Limits

**KEY IDEA:** There are often algebraic tricks we can use to compute a limit if the limit laws fail. Here is a summary of them:

1. Simplify the expression completely first.
2. Factor and Cancel
3. Multiply by the conjugate

Here are illustrative examples:

1. Compute the limit, if it exists:

$$\lim_{x\to 4} \frac{x^2-4x}{x^2-3x-4} = \lim_{x\to 4} \frac{x(x-4)}{(x-4)(x+1)} = \frac{4}{5}$$

   *IDEA: Factor and Cancel*

2. Along the same lines:

$$\lim_{x\to -1} \frac{x^2-4x}{x^2-3x-4} = \lim_{x\to -1} \frac{x(x-4)}{(x-4)(x+1)}$$

   Substituting $-1$ into the numerator gives us $(-1)(-5) = 5 \neq 0$. Since the fraction approaches $5/0$, the limit DNE (in fact, the $y-$values are going to $\pm\infty$

3. Compute the limit, if it exists:

$$\lim_{x\to 0} \frac{\sqrt{3+x}-\sqrt{3}}{x} \cdot \frac{\sqrt{3+x}+\sqrt{3}}{\sqrt{3+x}+\sqrt{3}} = \lim_{x\to 0} \frac{3+x-3}{x(\sqrt{3+x}+\sqrt{3})} = \frac{1}{2\sqrt{3}}$$

   *IDEA: Multiply by the conjugate*

4. An example of the Squeeze Theorem: Show that

$$\lim_{x\to 0} \sqrt{x^3+x^2}\,\sin\left(\frac{\pi}{x}\right) = 0$$

   SOLUTION: We know that the sine function is always less than 1. Therefore:

$$\sqrt{x^3+x^2}\,\sin\left(\frac{\pi}{x}\right) \leq \sqrt{x^3+x^2}\cdot 1$$

   Similarly, the sine function is always bigger than $-1$. Therefore,

$$\sqrt{x^3+x^2}(-1) \leq \sqrt{x^3+x^2}\,\sin\left(\frac{\pi}{x}\right) \leq \sqrt{x^3+x^2}\cdot 1$$

   Now we have our upper and lower functions. Each of them have a limit of zero at $x=0$, therefore (by the Squeeze Theorem) so does our function.