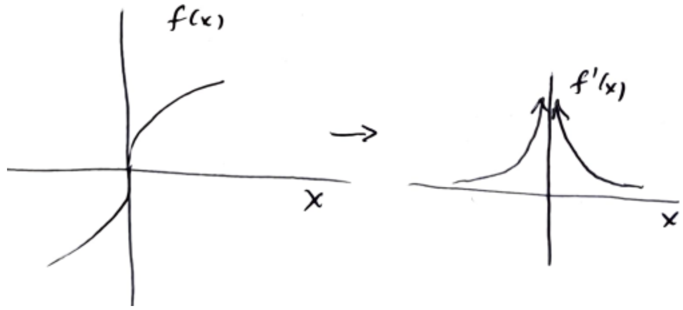
Math 125 Exam 2, Sample 1

Show all your work! Answers with no work may get no credit. No calculators or notes allowed.

1. Given the graph of $f(x)$ below, sketch a graph of $f'(x)$.



2. Differentiate:

   (a) $\left(\frac{\cos(x)}{\sqrt{x}}\right)' = \frac{-\sin(x)\sqrt{x} - \cos(x)(1/2)x^{-1/2}}{x}$

   (b) $\left(\sqrt{x+2^x}\right)' = \frac{1}{2}(x^2+2^x)^{-1/2}(2x+2^x\ln(2))$

   (c) $(\log_3(x)\cdot\tan(x))' = \frac{1}{x\ln(3)}\cdot\tan(x) + \log_3(x)\sec^2(x)$

3. Derive the formula for the derivative of $\tan(x)$ by first writing it in terms of sine and cosine.

$$(\tan(x))' = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

4. Find an equation for the tangent line to $\ln(x) + y^2 = xy$ at the point $(1,1)$.

   The point is given, we just need the slope- implicitly differentiate, then substitute $x = 1, y = 1$ and solve for y':

$$\frac{1}{x} + 2yy' = y + xy' \quad \Rightarrow \quad 1 + 2y' = 1 + y' \quad \Rightarrow \quad y' = 0$$

   This is a horizontal line $y = 1$.

5. Find an equation for the tangent line to $y = \tan^{-1}(x+1)$ at the point $(0, \pi/4)$.

   The point is given, just find the slope using the derivative:

$$y' = \frac{1}{1+(x+1)^2}\cdot 1 \quad \Rightarrow \quad y'(0) = \frac{1}{2} \quad \Rightarrow \quad y - \pi/4 = \frac{1}{2}(x-0)$$

6. Differentiate: $y = x^{x^2}$

   Use logarithmic differentiation:

$$\ln(y) = ln(x^{x^2}) = x^2\ln(x) \quad \Rightarrow \quad \frac{y'}{y} = 2x\ln(x) + x^2\cdot\frac{1}{x}$$

   Therefore, $y' = x^{x^2}(2x\ln(x) + x)$

7. A chemist has a 300 mg sample of radioactive isotope. After 4 days, there is 75 mg remaining. Find a formula for the amount of isotope she has after t days. What is the half-life of the element?

   Model equation is $m(t) = m_0 e^{kt}$. We're given $m_0 = 300$, and when $t = 4$, $m = 75$. Therefore, we can solve for k:

$$75 = 300e^{4k} \quad \Rightarrow \quad \frac{1}{4} = e^{4k} \quad \Rightarrow \quad -\ln(4) = 4k \quad \Rightarrow \quad k = \frac{-\ln(4)}{4} = \frac{-2\ln(2)}{4} = \frac{-\ln(2)}{2}$$

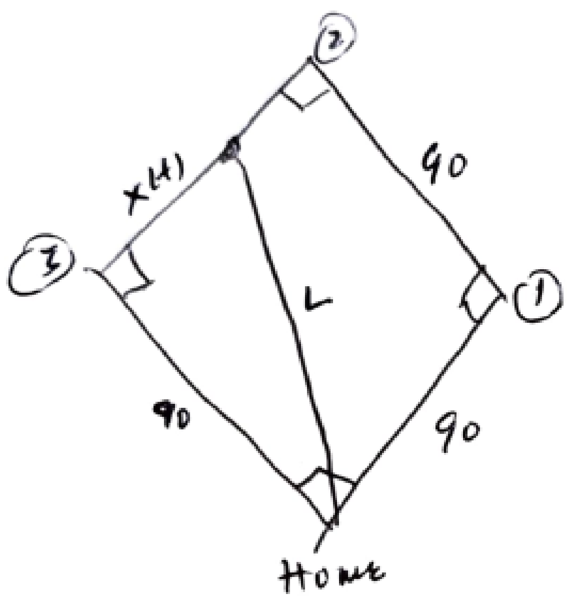
   Now the model equation is $m(t) = 300e^{-(\ln(2)/2)t}$. To find the half-life, solve

$$150 = 300e^{-(\ln(2)/2)t} \quad \Rightarrow \quad \frac{1}{2} = e^{-(\ln(2)/2)t} \quad \Rightarrow \quad -\ln(2) = -\frac{\ln(2)}{2}t \quad \Rightarrow \quad t = 2$$

   Therefore, the half-life is 2 days.

8. A baseball player is running from second base to third base at a rate of 20 feet per second. How fast is the distance of the player from home plate decreasing when he is half way between second and third?

   Recall that a baseball diamond is a square of dimensions 90 feet by 90 feet, and home plate and second base are on opposite corners of that square.



   See the sketch. From it, we get the Pythagorean theorem:

$$(x(t))^2 + 90^2 = L(t)^2 \quad \Rightarrow \quad 2x(t)\frac{dx}{dt} = 2L\frac{dL}{dt}$$

   so that

$$\frac{dL}{dt} = \frac{x}{L}\frac{dx}{dt}$$

   We know that $x = 45$ and $dx/dt = -20$, and $L = \sqrt{45^2 + 90^2}$. Putting these numbers in give us approximately -8.944 (without a calculator, you can leave your answer unsimplified as

$$\frac{dL}{dt} = \frac{45}{\sqrt{45^2 + 90^2}} \cdot (-20)$$

9. Use differentials to approximate $\sqrt{99.8}$ (not on this exam)