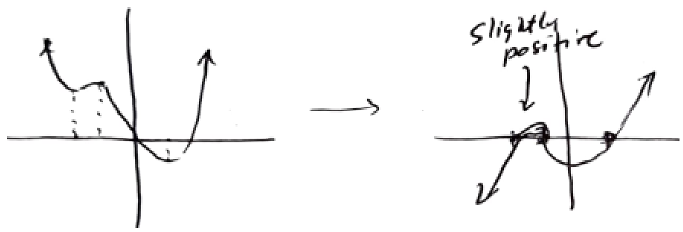
Math 125 Exam 2, Sample 2

Show all your work! Answers with no work may get no credit. No calculators or notes allowed.

1. Given the graph of $f(x)$ below, sketch a graph of $f'(x)$.



2. Differentiate:

   (a) $(\sec(\sqrt[5]{x}))' = \sec(\sqrt[5]{x})\tan(\sqrt[5]{x}) \cdot \dfrac{1}{5}x^{-4/5}$

   (b) $((x^2 + \log_3(|x|))^5)' = 5(x^2 + \log_3|x|)^4 \cdot \left(2x + \dfrac{1}{x\ln(3)}\right)$

   NOTE: In class, we showed that $(\ln|x|)' = \frac{1}{x}$, and this works for other bases as well- In this case, $(\log_3|x|)' = \frac{1}{x\ln(3)}$.

   (c) $\left(\dfrac{x^2+1}{5^x}\right)' = \dfrac{2x \cdot 5^x - (x^2+1)5^x\ln(5)}{(5^x)^2}$

3. Derive the formula for the derivative of $\sec(x)$ by first writing it in terms of sine and cosine.

$$(\sec(x))' = \left(\frac{1}{\cos(x)}\right)' = ((\cos(x))^{-1})' = -1(\cos(x))^{-2}(-\sin(x)) = \frac{\sin(x)}{\cos^2(x)} = \frac{1}{\cos(x)}\frac{\sin(x)}{\cos(x)} = \sec(x)\tan(x)$$

4. Find an equation for the tangent line to $x + y + x^2y^2 = 5$ at the point $(2, -1)$.

$$1 + y' + (2xy^2 + x^2 \cdot 2yy') = 0 \quad \overset{x=2,\,y=-1}{\Rightarrow} \quad 1 + y' + 4 - 8y' = 0 \quad \Rightarrow \quad 5 - 7y' = 0 \quad \Rightarrow \quad y' = \frac{5}{7}$$

so the tangent line is:

$$y + 1 = \frac{5}{7}(x - 2)$$

5. Find an equation for the tangent line to $y = \sin^{-1}(x)$ at the point $(-1/2, -\pi/6)$.

$$y' = \frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \quad y'(-1/2) = \frac{1}{\sqrt{1-\frac{1}{4}}} = \frac{2}{\sqrt{3}}$$

The line is $y + \frac{\pi}{6} = \frac{2}{\sqrt{3}}\left(x + \frac{1}{2}\right)$

6. Differentiate: $y = (\cos(x))^{\ln(x)}$

   Use logarithmic differentiation- take the log of both sides:

$$\ln(y) = \ln\left(\cos(x)^{\ln(x)}\right) = \ln(x)\ln(cos(x))$$

Now differentiate:

$$\frac{1}{y}y' = \frac{1}{x}\ln(\cos(x)) + \ln(x)\cdot\frac{1}{\cos(x)}(-\sin(x))$$

Lastly, multiply both sides by y (substitute the expression in x):

$$y = \cos(x)^{\ln(x)}\left(\frac{1}{x}\ln(\cos(x)) + \ln(x)\cdot\frac{1}{\cos(x)}(-\sin(x))\right)$$

7. If $f(x) = 2x + \mathrm{e}^x$, find the equation of the tangent line **to the inverse** of f at $(1, 0)$. HINT: Do not try to compute f^{-1} algebraically.

   Solution: Notice that the point $(0, 1)$ goes through the graph of $f(x) = 2x + \mathrm{e}^x$, and $f'(0) = 2 + \mathrm{e}^0 = 3$. Therefore, $(1, 0)$ goes through the graph of the inverse, and the value of the derivative $(f^{-1})'(1) = \frac{1}{3}$. The equation of the line is:

$$y - 0 = \frac{1}{3}(x - 1)$$

8. A population of rabbits start out at 50 rabbits. After three years there are 200 rabbits. What is the doubling time for the population? When will there be 500 rabbits?

   The model equation is $P(t) = P_0\mathrm{e}^{rt}$, and P_0 is given as 50. Using the data point $(3, 200)$, we can solve for the value of k:

$$200 = 50\mathrm{e}^{3k} \quad\Rightarrow\quad 4 = \mathrm{e}^{3k} \quad\Rightarrow\quad \ln(4) = 3k \quad\Rightarrow\quad k = \frac{\ln(4)}{3}$$

   This gives the model equation as $P(t) = 50\mathrm{e}^{(\ln(4)/3)t}$. For the second part, double the population is 100, so solve:

$$100 = 50\mathrm{e}^{(\ln(4)/3)t} \quad\Rightarrow\quad 2 = \mathrm{e}^{(\ln(4)/3)t} \quad\Rightarrow\quad \ln(2) = \frac{\ln(4)}{3}t \quad\Rightarrow\quad t = \frac{\ln(2)\cdot 3}{\ln(4)}$$

   It would be OK to stop here, but we can simplify, since $4 = 2^2$:

$$t = \frac{\ln(2)\cdot 3}{2\ln(2)} = \frac{3}{2}$$

   Similarly, for 500:

$$500 = 50\mathrm{e}^{(\ln(4)/3)t} \quad\Rightarrow\quad 10 = \mathrm{e}^{(\ln(4)/3)t} \quad\Rightarrow\quad \ln(10) = \frac{\ln(4)}{3}t \quad\Rightarrow\quad t = \frac{\ln(10)\cdot 3}{\ln(4)}$$

9. A light is on the ground and points at a building 20 meters away. A man who is 2 meters tall starts at the light and runs toward the building in a straight path at the rate of 5 meters per second. How fast is the top of his shadow moving down the wall when the man is 10 meters from the light?

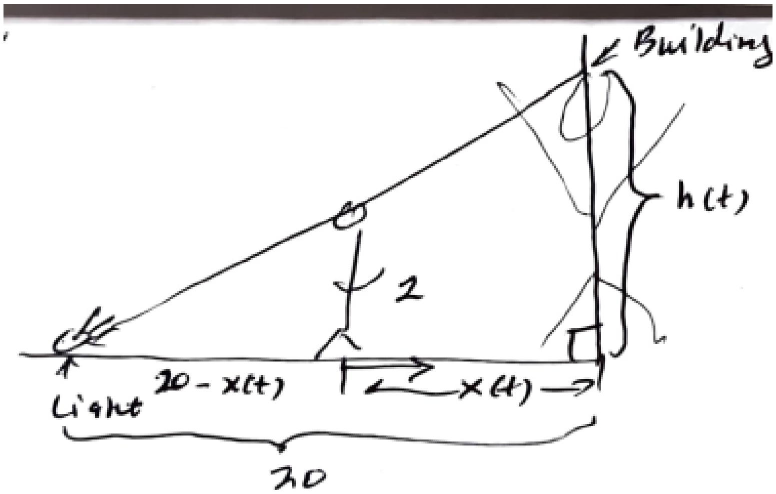
   See the sketch. From similar triangles, we get

$$\frac{h(t)}{2} = \frac{20}{20 - x(t)} \quad\Rightarrow\quad h(t) = 40(20 - x(t))^{-1}$$

   Therefore,

$$\frac{dh}{dt} = 40\cdot(-(20 - x(t))^{-2})\cdot -\frac{dx}{dt}$$

   Substitute $x = 10$ and $dx/dt = 5$, and we should get $dh/dt = -2$ m/s.


Building
h(t)
2
Light
20 - x(t)
x(t)
20