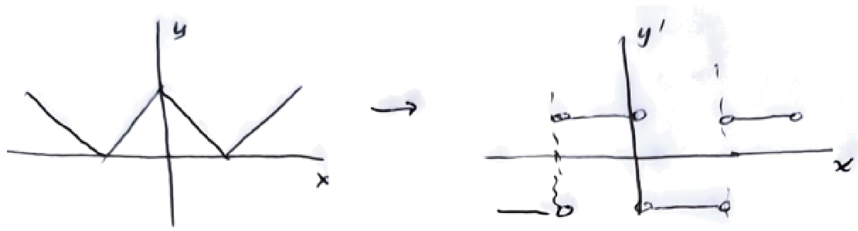
Math 125 Exam 2, Sample 3

Show all your work! Answers with no work may get no credit. No calculators or notes allowed.

1. Given the graph of $f(x)$ below, sketch a graph of $f'(x)$.



2. Differentiate:

   (a) $y = \log_2(x)\sec(x)$

$$y' = \frac{1}{x\ln(2)}\sec(x) + \log_2(x)\sec(x)\tan(x)$$

   (b) $y = \sqrt{\sin(\sqrt{x})}$

$$y' = \frac{1}{2}(\sin(\sqrt{x}))^{-1/2} \cdot \cos(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2}$$

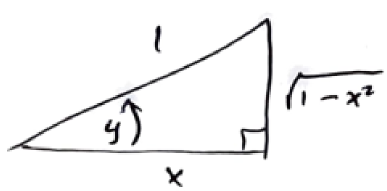
   (c) $y = x3^{-1/x}$

$$y' = 3^{-1/x} + x \cdot 3^{-1/x}\ln(3) \cdot \frac{1}{x^2}$$

   (d) $y = \frac{2x+5}{5^x}$

$$y' = \frac{2 \cdot 5^x - (2x+5) \cdot 5^x \ln(5)}{(5^x)^2}$$

3. Derive the formula for the derivative of $\cos^{-1}(x)$. We started by: $y = \cos^{-1}(x)$, which is $\cos(y) = x$. We can then differentiate implicitly- Fill in the rest of the argument.



SOLUTION: See the sketch (from $\cos(y) = x$). Then

$$-\sin(y)\frac{dy}{dt} = 1 \quad \Rightarrow \quad \frac{-1}{\sin(y)} = \frac{-1}{\sqrt{1-x^2}}$$

4. Find the equation of the tangent line to $\sqrt{y} + xy^2 = 5$ at the point $(4, 1)$.

SOLUTION:

$$\frac{1}{2}y^{-1/2}\frac{dy}{dx} + y^2 + x \cdot 2yy' = 0$$

Substitute $x = 4$, $y = 1$:

$$\frac{1}{2}y' + 1 + 8y' = 0 \quad \Rightarrow \quad \frac{17}{2}y' = -1 \quad \Rightarrow \quad y' = -\frac{2}{17}$$

Therefore, the equation is: $y - 1 = \frac{-2}{17}(x - 4)$.

5. Differentiate: $y = x^{\cos(x)}$

   Use logarithmic differentiation by taking the log of both sides, simplify, then differentiate.

$$\ln(y) = \ln(x^{\cos(x)}) \quad \Rightarrow \quad \ln(y) = \cos(x) \cdot \ln(x)$$

   Differentiate:

$$\frac{1}{y}y' = -\sin(x)\ln(x) + \cos(x) \cdot \frac{1}{x}$$

   Multiply both sides by y (back substitute the expression for y):

$$y' = x^{\cos(x)} \left(-\sin(x)\ln(x) + \frac{\cos(x)}{x} \right)$$

6. True or False, and explain:

   (a) The derivative of a polynomial is a polynomial.
   True. From the power rule, the order of the polynomial decreases by 1 each time.

   (b) If f is differentiable, then $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$
   True, by the chain rule.

   (c) The derivative of $y = \sec^{-1}(x)$ is the derivative of $y = \cos(x)$.
   False. $\sec^{-1}(x)$ is the inverse secant function, not the reciprocal of the secant.

   (d) $\frac{d}{dx}(10^x) = x10^{x-1}$
   False. Wrong rule- We needed the exponential function rule, not the power rule.

   (e) If $y = \ln|x|$, then $y' = \frac{1}{x}$
   True. We showed this in class, and it's not hard to prove (we ended up putting this in the table on the board).

   (f) The equation of the tangent line to $y = x^2$ at $(1, 1)$ is: $y - 1 = 2x(x - 1)$
   False. The expression $2x$ is a formula for the slope of the tangent line, not the slope itself. The slope of the tangent line is 2.

   (g) If $y = e^2$, then $y' = 2e$
   False. e^2 is a constant, so the derivative is 0.

   (h) If $y = ax + b$, then $\frac{dy}{da} = x$
   True. Differentiating both sides by a will give the equation shown.

7. If $f(x) = 2x + e^x$, find the equation of the tangent line **to the inverse** of f at $(1, 0)$. HINT: Do not try to compute f^{-1} algebraically.

   Oops- this was the same question from Sample Exam 2.

8. Radium-226 has a half-life of 1600 years. How long does it take for 18 grams of Radium-226 to decay to leave a total of 2.25 grams? (You can use a calculator for this practice problem- the numbers will work out nicely on the exam).

   SOLUTION: Using the model equation, $m(t) = m_0 e^{kt}$, we have our initial value, $m_0 = 18$, so the half-life of 1600 years gives us a way to solve for k:

$$\frac{m_0}{2} = m_0 e^{1600k} \quad \Rightarrow \quad k = -\frac{\ln(2)}{1600}$$

   Now our model equation is fully determined,

$$m(t) = 18e^{-(\ln(2)/1600)t}$$

Finally, solve for t if $m = 2.25$:

$$2.25 = 18\mathrm{e}^{-(\ln(2)/1600)t} \quad \Rightarrow \quad 0.125 = \mathrm{e}^{-(\ln(2)/1600)t}$$

Taking the logs (and using your calculator):

$$\ln(0.125) = -\frac{\ln(2)}{1600}t \quad \Rightarrow \quad t = 4800$$

So it takes 4800 years for a mass of 18 grams to reduce to 2.25 grams.

9. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm. The surface area of a sphere is given by $SA = 4\pi r^2$, where r is the radius.

   SOLUTION: The surface area formula is in terms of the radius instead of the diamater D, where $D = 2r$. Substituting $r = D/2$, we get

$$A = 4\pi \left(\frac{D(t)}{2}\right)^2 = \pi D(t)^2$$

   Now,

$$\frac{dA}{dt} = 2\pi D(t) \cdot \frac{dD}{dt}$$

   And substitute $D = 10$, $dA/dt = -1$:

$$\frac{dD}{dt} = \frac{-1}{20\pi}$$