Math 125
Fall 2019

EXAM I

NAME: Solutions

Show all your work! Be sure to show your algebra- an answer with no work will receive no credit. You may not use rules of calculus that we have not discussed to justify an answer.

1. (10 points)

   (a) State the definition of the derivative of f at $x = a$:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

   (b) Finish the definition: f is continuous at $x = a$ if: $\lim_{x \to a} f(x) = f(a)$

2. (10 pts) Find $h'(1)$ using the definition, if $h(t) = \sqrt{1 + 2t}$.

$$\lim_{h \to 0} \frac{\sqrt{1+2(1+h)} - \sqrt{1+2}}{h} = \lim_{h \to 0} \frac{\sqrt{3+2h} - \sqrt{3}}{h} \cdot \frac{\sqrt{3+2h}+\sqrt{3}}{\sqrt{3+2h}+\sqrt{3}}$$

$$= \lim_{h \to 0} \frac{\cancel{3} + 2h - \cancel{3}}{h(\sqrt{3+2h}+\sqrt{3})} = \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}(\sqrt{3+2h}+\sqrt{3})} = \frac{2}{\sqrt{3}+\sqrt{3}} = \frac{2}{2\sqrt{3}}$$

$$= \boxed{\frac{1}{\sqrt{3}}}$$

3. (10 points) Find the values of a and b so that $f(x)$ is continuous for all x:

$$f(x) = \begin{cases} 4a - x^2, & \text{if } x < 3 \\ b, & \text{if } x = 3 \\ 3 - ax, & \text{if } x > 3 \end{cases}$$

(Be sure your answer shows that you understand the definition of continuity)

(1) Does $f(3)$ exist? $f(3) = b$ ✓

(2) $\lim_{x \to 3^-} 4a - x^2 = 4a - 9$; $\lim_{x \to 3^+} 3 - ax = 3 - 3a$

We need $4a - 9 = 3 - 3a$, so $7a = 12$, or $a = 12/7$

(3) For (1) = (2), $b = 3 - 3(12/7) = \frac{21-36}{7} = -15/7$

4. (10 points) Where is the function continuous?

$$f(x) = \frac{1}{\sqrt{x^2 - 9}}$$

f is cont. on the domain of f,

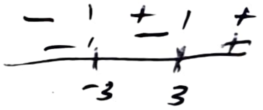
which is $(-\infty, -3) \cup (3, \infty)$

or $x < -3$, $x > 3$

$x^2 - 9 > 0$

or

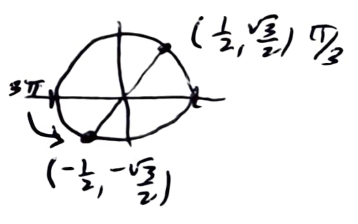
$(x+3)(x-3) > 0$



5. (10 points) Give the exact value:

(a) $\cos(10\pi/3)$ $= \boxed{-1/2}$

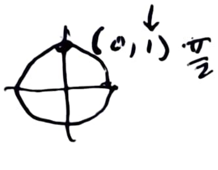
Note: $9\pi/3 = 3\pi$, so this is $\pi/3$ more



(b) $\cot(\pi/4) = \frac{\cos(\pi/4)}{\sin(\pi/4)} = \boxed{1}$

(c) $\arcsin(1) = \theta$, where $\sin(\theta) = 1$

$\theta = \boxed{\frac{\pi}{2}}$



6. (15 pts) Find the equation of the tangent line to $y = \frac{2}{1-3x}$ at $x = 0$.

$f(0) = \frac{2}{1} = 2$

Point: $(0, 2)$

$$y' = \lim_{h \to 0} \frac{\frac{2}{1-3(0+h)} - \frac{2}{1-0}}{h} = \lim_{h \to 0} \frac{\frac{2}{1-3h} - 2}{h}$$

$$= \lim_{h \to 0} \frac{2 - \overset{-2+6h}{2(1-3h)}}{1-3h} \cdot \frac{1}{h} = \lim_{h \to 0} \frac{6h}{h(1-3h)} = 6$$

Eqn: $y - 2 = 6(x - 0)$ or $y = 6x + 2$.

7. (15 points) Find each limit, if it exists.

(a) $\lim_{x \to -1} \frac{x^2 - 4x}{x^2 - 3x - 4}$ = [redacted] "$\frac{5}{0}$"

Goes to $\pm\infty$ as $x \to -1$ from left/right.

When $x < 0$, $x = -\sqrt{x^2}$

(b) $\lim_{x \to -\infty} \frac{\sqrt{4x^2+1} \,/x}{x+1 \,/x} = \lim_{x \to -\infty} \frac{-\sqrt{\frac{4x^2+1}{x^2}}}{1 + 1/x} = -2$

8. (20 pts) Short Answer:

(a) Show that there must be at least one real solution to $\underbrace{x^5 - x^3 + 3x - 5}_{f(x)} = 0$.

$f(0) = -5$

$\boxed{\begin{array}{l} f(1) = -2 \\ f(2) = 32 - 8 + 6 - 5 = 25 \end{array}}$

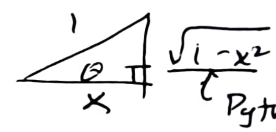
Since $f(1) < 0$, $f(2) > 0$,

we must have a c in $(1, 2)$

such that $f(c) = 0$ (IVT).

(b) Simplify: $\sin(\cos^{-1}(x))$.

Let $\theta = \cos^{-1}(x)$ so that $\cos(\theta) = \frac{x}{1} \begin{array}{l}\leftarrow \text{adj} \\ \leftarrow \text{hyp}\end{array}$



Pythag Thm

$\Rightarrow \sin(\cos^{-1}(x)) = \sin(\theta) = \sqrt{1-x^2}$

(c) If $2x + 1 \leq f(x) \leq x^2 + 2$ for all x, then what is $\lim_{x \to 1} f(x)$? (Give a short justification)

As $x \to 1$, $2x+1 \to 3$ and $x^2 + 2 \to 0$

$\Rightarrow f(x)$ must also go to 3 by the squeeze theorem.

(d) Compute $\log_3(45) - \log_3(10) + \log_3(54)$

$$\log_3\left(\frac{\overset{3^2}{\cancel{9}} \cdot \cancel{5} \cdot \overset{3}{\cancel{6}} \cdot \overset{3^2}{\cancel{9}}}{\cancel{5} \cdot \cancel{2}}\right) = \log_3(3^5) = 5$$