Math 125 Exam 2 Review: 2.8 and Ch 3

This exam covers sections 2.8 and all of chapter 3, except for 3.7 and 3.11.

From 2.8, recall the definition of the derivative and be able to sketch the derivative given the graph of the function. In sections 3.1-3.6, we learned all of the derivative formulas (summarized in the table- be sure to know them!). In sections 3.8-3.10, we look at applying the derivative in different situations.

Vocabulary/Techniques:

- Be sure you distinguish between:

  $$a^x \text{ or } a^{f(x)} \qquad x^a \text{ or } (f(x))^a \qquad f(x)^{g(x)}$$

  How do we differentiate each expression?

  - $a^{f(x)}$ is differentiated as: $f'(x)a^{f(x)}\ln(a)$ (this is the general exponential)
  - $(f(x))^a$ is differentiated as: $a(f(x))^{a-1}f'(x)$ (The power rule)
  - $f(x)^{g(x)}$ is differentiated using logarithmic differentiation, $\ln(y) = g(x)\ln(f(x))$

- Know the definition of "differentiable".

- Understand the relationship between "differentiable" and "continuous".

- Implicit Differentiation: A technique where we are given an equation with x, y. We treat y as a function of x, and differentiate without explicitly solving for y first.

  Example: $x^2y + \sqrt{xy} = 6x \to (2xy + x^2y') + \frac{1}{2}(xy)^{-\frac{1}{2}}(y + xy') = 6$

- Logarithmic Differentiation: A technique where we apply the logarithm to $y = f(x)$ before differentiating. Used for taking the derivative of complicated expressions, and needed for taking the derivative of $f(x)^{g(x)}$.

  Example: $y = x^x \to \ln(y) = x\ln(x) \to \frac{1}{y}y' = \ln(x) + 1 \to ...$ etc

- Differentiation of Inverses: If we know the derivative of $f(x)$, then we can determine the derivative of $f^{-1}(x)$. This technique was used to find derivatives of the inverse trig functions, for example:

  $$y = f^{-1}(x) \quad \Rightarrow \quad f(y) = x \quad \Rightarrow \quad f'(y)y' = 1$$

  And then we converted the expression back to x. You could also express the relationship as: If $f(a) = b$, then

  $$\frac{d}{dx}\left(f^{-1}(b)\right) = \frac{1}{f'(a)}$$

  Alternatively, we say that if (a, b) is on the graph of f and $f'(a) = m$, then we know that (b, a) is on the graph of f^{-1}, and $\frac{df^{-1}}{dx}(b) = \frac{1}{m}$.

  NOTE: This is NOT the same as the derivative of the reciprocal $(f(x))^{-1} = \frac{1}{f(x)}$, which is computed using the reciprocal rule (See our derivative table).

- We also have an alternative version of the Chain Rule that may be useful in certain cases.

  For example, if Volume is a function of Radius, Radius is a function of Pressure, Pressure is a function of time, then we can find the rate of change of Volume in terms of the Radius, or the Pressure, or the time. Respectively, this is:

  $$\frac{dV}{dR}, \qquad \frac{dV}{dP} = \frac{dV}{dR}\cdot\frac{dR}{dP}, \qquad \frac{dV}{dt} = \frac{dV}{dR}\cdot\frac{dR}{dP}\cdot\frac{dP}{dt}$$

  In fact, we could also find things like dR/dV, dP/dR, and so on because of the relationship between the derivative of a function and its inverse: $dx/dy = 1/(dy/dx)$.

- Things that come up in the inverse trig stuff: Be able to simplify expressions like $\tan(\cos^{-1}(x))$, $\sin(\tan^{-1}(x))$, etc. using an appropriate right triangle.
- Remember the logarithm rules:
  1. $A = \mathrm{e}^{\ln(A)}$ for any $A > 0$.
  2. $\log(ab) = \log(a) + \log(b)$
  3. $\log(a/b) = \log(a) - \log(b)$
  4. $\log(a^b) = b\log(a)$
- It is best to use parentheses for the argument of a function. For example, $\sin(x)$ is more clear than $\sin x$ (Our book likes to drop the parentheses, but it can be confusing in some situations).
- Always simplify BEFORE differentiating.

  Example: With $y = x\sqrt{x}$, don't use the product rule- First rewrite as $y = x^{3/2}$

  Example: With $y = \frac{3x^2+5x+1}{\sqrt{x}}$, don't use the quotient rule-

  First rewrite as $y = 3x^{3/2} + 5x^{1/2} + x^{-1/2}$
- Exponential Growth and Decay model (3.8):

  The growth and decay model is $y' = ry$. The function that satisfies this differential equation is: $y(t) = P_0\mathrm{e}^{rt}$ (or "Pert").

  Be able to find half life or doubling time, or use information about the rate of change to find r and perhaps P_0. Be able to apply the model to radioactive decay, bacterial growth, or growth of a financial investment (with continuous compounding).
- Related Rates (3.9)

  Formulas we should be familiar with: Area, perimeter, circumference. We should also know Pythagorean Theorem and Similar Triangles.

  Formulas for spheres and cones will be provided.

  To write and solve a related rates problem, we generally:
  - Draw a sketch, label variables.
  - Find a relationship between the variables/constants.
    This is where we may need a geometric relationship.
  - Differentiate and solve.
- (3.10) Linear approximations and differentials:

$$\begin{array}{rll} L(x) & = f(a) + f'(a)(x-a) & \text{Linearization of } f \text{ at } a \\ \Delta y & = f(x+\Delta x) - f(x) & \text{Actual change of } f \\ dy & = f'(x)\,dx & \text{The differential - Used to approximate } \Delta y \end{array}$$

  The *relative* change in y is $\Delta y/y$ and was approximated by dy/y.