Exam 2 Review: 2.9-3.8

This portion of the course covered the bulk of the formulas for differentiation, together with a few definitions and techniques. Remember that we also left half of 2.9 for this section.

The following tables summarize the rules that we've had:

$f(x)$	$f'(x)$
c	0
x^n	nx^{n-1}
a^x	$a^x \ln(a)$
e^x	e^x
$\log_a(x)$	$\frac{1}{x\ln(a)}$
$\ln(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\cot(x)$	$-\csc^2(x)$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$

$f(x)$	$f'(x)$
cf	cf'
$f \pm g$	$f' \pm g'$
$f \cdot g$	$f'g + fg'$
$\frac{f}{g}$	$\frac{f'g-fg'}{g^2}$
$f(g(x))$	$f'(g(x))g'(x)$
$f(x)^{g(x)}$	log diff

Vocabulary/Techniques:

- Differentiable: A function f is differentiable at $x = a$ if the limit exists. (Remember the definition of the derivative?) A function f is differentiable on (a, b) if it is differentiable at each point in the interval. Graphically, this means the function is smooth with no vertical tangents.
- Implicit Differentiation: A technique where we are given an equation with x, y. We treat y as a function of x, and differentiate without explicitly solving for y first.

  Example: $x^2y + \sqrt{xy} = 6x \to 2xy + x^2y' + \frac{1}{2}(xy)^{-\frac{1}{2}}(y + xy') = 6$
- Logarithmic Differentiation: A technique where we apply the logarithm to $y = f(x)$ before differentiating. Used for taking the derivative of complicated expressions, and needed for taking the derivative of $f(x)^{g(x)}$.

  Example: $y = x^x \to \ln(y) = x\ln(x) \to \frac{1}{y}y' = \ln(x) + 1 \to ...$ etc
- Differentiation of Inverses: If we know the derivative of $f(x)$, then we can determine the derivative of $f^{-1}(x)$. This technique was used to find derivatives of the inverse trig functions, for example:

  $y = f^{-1}(x) \Rightarrow f(y) = x \Rightarrow f'(y)y' = 1$ From this, we could write:

  $$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$$

NOTE: This is NOT the same as the derivative of $(f(x))^{-1} = \frac{1}{f(x)}$, which is

$$\frac{d}{dx}\left((f(x))^{-1}\right) = -\left(f(x)\right)^{-2} f'(x) = \frac{-f'(x)}{(f(x))^2}$$

Other Notes:

- The words: "Rate of Change" are translated as "the derivative"
- If we have: Volume is a function of Radius, Radius is a function of Pressure, Pressure is a function of time, then we can find the rate of change of Volume in terms of the Radius, or the Pressure, or the time. Respectively, this is:

$$\frac{dV}{dR}, \quad \frac{dV}{dP} = \frac{dV}{dR} \cdot \frac{dR}{dP}, \quad \frac{dV}{dt} = \frac{dV}{dR} \cdot \frac{dR}{dP} \cdot \frac{dP}{dt}$$

  This is one way of interpreting the Chain Rule.
- Remember the logarithm rules:
  1. $A = \mathrm{e}^{\ln(A)}$ for any $A > 0$.
  2. $\log(ab) = \log(a) + \log(b)$
  3. $\log(a/b) = \log(a) - \log(b)$
  4. $\log(a^b) = b\log(a)$
- Always simplify BEFORE differentiating. Example: To differentiate $y = x\sqrt{x}$, first rewrite as $y = x^{3/2}$

Exam 2 Review Questions

1. True or False, and explain:

   (a) The derivative of a polynomial is a polynomial.

   (b) If f is differentiable, then $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$

   (c) The derivative of $y = \sec^{-1}(x)$ is the derivative of $y = \cos(x)$.

   (d) $\frac{d}{dx}(10^x) = x10^{x-1}$

   (e) The equation of the tangent line to $y = x^2$ at $(1, 1)$ is:

   $$y - 1 = 2x(x - 1)$$

   (f) If $y = e^2$, then $y' = 2e$

   (g) If $y = ax + b$, then $\frac{dy}{da} = x$

2. Find the equation of the tangent line to $x^3 + y^3 = 3xy$ at the point $(\frac{3}{2}, \frac{3}{2})$.

3. If $f(0) = 0$, and $f'(0) = 2$, find the derivative of $f(f(f(f(x))))$ at $x = 0$.

4. Derive the formula for the derivative of $y = \cos^{-1}(x)$ using implicit differentiation.

5. Find the equation of the tangent line to $\sqrt{y} + xy^2 = 5$ at the point $(4, 1)$.

6. If $s^2t + t^3 = 1$, find $\frac{dt}{ds}$ and $\frac{ds}{dt}$.

7. If $y = x^3 - 2$ and $x = 3z^2 + 5$, then find $\frac{dy}{dz}$.

8. A space traveler is moving from left to right along the curve $y = x^2$. When she shuts off the engines, she will go off along the tangent line at that point. At what point should she shut off the engines in order to reach the point $(4, 15)$?

9. A particle moves in the plane according to the law $x = t^2+2t$, $y = 2t^3-6t$. Find the slope of the tangent line when $t = 0$.

10. Find the coordinates of the point on the curve $y = (x - 2)^2$ at which the tangent line is perpendicular to the line $2x - y + 2 = 0$.

11. For what value(s) of A, B, C does the polynomial $y = Ax^2 + Bx + C$ satisfy the differential equation:

    $$y'' + y' - 2y = x^2$$

    Hint: If $ax^2 + bx + c = 0$ for ALL x, then $a = 0, b = 0, c = 0$.

12. If $V = \sin(w)$, $w = \sqrt{u}$, $u = t^2 + 3t$, compute: The rate of change of V with respect to w, the rate of change of V with respect to u, and the rate of change of V with respect to t.

13. Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

14. Differentiate. You may assume that y is a function of x, if not already defined explicitly.

    (a) $\sqrt{2xy} + xy^3 = 5$ (and solve for $\frac{dy}{dx}$)

    (b) $y = \sqrt{x^2 + \sin(x)}$

    (c) $y = \mathrm{e}^{\cos(x)} + \sin(5^x)$

    (d) $y = \cot(3x^2 + 5)$

    (e) $y = x^{\cos(x)}$

    (f) $y = \sqrt{\sin(\sqrt{x})}$

    (g) $\sqrt{x} + \sqrt[3]{y} = 1$

    (h) $x \tan(y) = y - 1$

    (i) $y = \sqrt{x}\, \mathrm{e}^{x^2} (x^2 + 1)^{10}$ (Hint: Logarithmic Diff)

    (j) $y = \sin^{-1}\left(\tan^{-1}(x)\right)$

    (k) $y = \ln|\csc(3x) + \cot(3x)|$

    (l) $y = \frac{-2}{\sqrt[4]{t^3}}$

    (m) $y = x3^{-1/x}$

    (n) $y = x \tan^{-1}(\sqrt{x})$

    (o) $y = \mathrm{e}^{2^{\mathrm{e}^x}}$

    (p) Let a be a positive constant. $y = x^a + a^x$

    (q) $x^y = y^x$

    (r) $y = \ln\left(\sqrt{\frac{3x+2}{3x-2}}\right)$