## Calculus I Review Solutions

1. Compare and contrast the three "Value Theorems" of the course. When you would typically use each.

   The three value theorems are the Intermediate, Mean and Extreme value theorems. The intermediate and extreme value theorems only require the function to be continuous on a closed interval. The mean value theorem additionally requires $f$ to be differentiable on the open interval.

   (a) We use the Mean Value Theorem for problems relating the derivative of an interior point to the values of $f$ at the endpoints. For example, if $f$ has two roots at $x = a$, $x = b$, then we know $f$ must have a critical point between $a$ and $b$. A common use is to substitute $f'(c)(b - a)$ in place of $f(b) - f(a)$.

   (b) The Intermediate Value Theorem says that for every $y-$value $w$ between $f(a)$ and $f(b)$, there is an $x$ in $[a, b]$ so that $f(x) = w$. We can use this to prove the existence of solutions to $f(x) = 0$, if $f(a)$ and $f(b)$ are different in sign.

   (c) The Extreme Value Theorem tells us when we can guarantee the existence of maximums and minimums, and tells us where they occur (at endpoints or a critical point).

2. List the three things we need to check to see if a function $f$ is continuous at $x = a$.

   (1) $f(a)$ exists, (2) $\lim_{x \to a} f(x)$ exists, and (3) Parts (1) and (2) have the same value.

3. Derive the formula for the derivative of $y = \sec^{-1}(x)$.

   First, write the expression as $\sec(y) = x$. Now draw a triangle with $y$ one of the angles. Since $\sec(y) = \frac{\text{hyp}}{\text{adj}}$, the hypotenuse is $x$, the side adjacent is 1 and the third side is $\sqrt{x^2 - 1}$.

   Now, differentiate $\sec(y) = x$:

   $$\sec(y)\tan(y)\frac{dy}{dx} = 1$$

   and $\sec(y) = x$, $\tan(y) = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{x^2-1}}{1}$ so

   $$x \cdot \sqrt{x^2 - 1} \cdot \frac{dy}{dx} = 1$$

   so

   $$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$$

4. Find the point on the parabola $x + y^2 = 0$ that is closest to the point $(0, -3)$.

   In general, the distance between a point $(a, b)$ and $(x, y)$ is

   $$d = \sqrt{(x - a)^2 + (y - b)^2}$$

but minimizing $d$ is equivalent to minimizing

$$D = (x-a)^2 + (y-b)^2$$

which is MUCH easier to differentiate. In this case,

$$D = (x-0)^2 + (y-(-3))^2 = x^2 + (y+3)^2$$

The equation of the parabola is used to get $D$ in terms of one variable. Since $x + y^2 = 0, x = -y^2$, so

$$D = (-y^2)^2 + (y+3)^2 = y^4 + (y+3)^2$$

Now we set the derivative equal to zero and solve. With the **hint from class**, we write:

$$4y^3 + 2y + 6 = 2(y+1)(2y^2 - 2y + 3)$$

From which we get that $y = -1$ is the only critical point (if you use the quadratic formula on $2y^2 - 2y + 3$, you'll get no real solutions).

Now, $x = -(-1)^2 = -1$. Is it a minimum? Yes, which you can get from looking at sign changes of the derivative.

5. Write the equation of the line tangent to $x = \sin(2y)$ at $x = 1$.

   We need a point and a slope. If $x = 1$, then

   $$1 = \sin(2y)$$

   so that $2y = \frac{\pi}{2}$, since $\sin(\pi/2) = 1$. Now, $y = \frac{\pi}{4}$. OK, so now we need a slope:

   $$1 = \cos(2y) \cdot 2\frac{dy}{dx}$$

   and the slope at $y = \frac{\pi}{4}$ is

   $$1 = \cos(\pi/2) \cdot 2\frac{dy}{dx} \Rightarrow 1 = 0$$

   This means that there is no slope- the tangent line is vertical. Therefore, the equation of the tangent line is $x = 1$.

6. For what values of $A, B, C$ will $y = Ax^2 + Bx + C$ satisfy the differential equation:

   $$\frac{1}{2}y'' - 2y' + y = 3x^2 + 2x + 1$$

   To start, we need expressions for $y, y', y''$ which we will then substitute into the equation:

   $$\begin{aligned} y &= Ax^2 + Bx + C \\ y' &= 2Ax + B \\ y'' &= 2A \end{aligned}$$

so we get:

$$\frac{1}{2}(2A) - 2\,(2Ax + B) + \Big(Ax^2 + Bx + C\Big) = 3x^2 + 2x + 1$$

Let's re-write this expression by combining the $x^2, x$ and constants:

$$Ax^2 + (B - 4A)x + (C + A - 2B) = 3x^2 + 2x + 1$$

From which we get:

$$\begin{aligned} A &= 3 \text{ from } x^2 \text{ term} \\ B - 4A &= 2 \text{ from } x \text{ term} \\ C + A - 2B &= 1 \text{ from constants} \end{aligned}$$

from which we get $A = 3, B = 14, C = 26$.

7. Compute the derivative of $y$ with respect to $x$:

   (a) $y = \sqrt[3]{2x+1}\sqrt[5]{3x-2}$
   Use logarithmic differentiation (its easier):

   $$\ln(y) = \frac{1}{3}\ln(2x+1) + \frac{1}{5}\ln(3x-2)$$

   so that

   $$\frac{1}{y}y' = \frac{1}{3}\cdot\frac{1}{2x+1}\cdot 2 + \frac{1}{5}\cdot\frac{1}{3x-2}\cdot 3$$

   Now,

   $$y' = \left(\sqrt[3]{2x+1}\sqrt[5]{3x-2}\right)\left(\frac{2}{3(2x+1)} + \frac{3}{5(3x-2)}\right)$$

   (b) $y = \frac{1}{1+u^2}$, where $u = \frac{1}{1+x^2}$
   In this case, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$, so

   $$\frac{dy}{du} = \frac{-2u}{(1+u^2)^2}, \quad \frac{du}{dx} = \frac{-2x}{(1+x^2)^2}$$

   so

   $$\frac{dy}{dx} = \frac{4ux}{(1+u^2)^2(1+x^2)^2}$$

   with $u = \frac{1}{1+x^2}$, which you can either state or explicitly substitute.

   (c) $\sqrt[3]{y} + \sqrt[3]{x} = 4xy$
   Implicit differentiation:

   $$\frac{1}{3}y^{-2/3}\frac{dy}{dx} + \frac{1}{3}x^{-2/3} = 4y + 4x\frac{dy}{dx}$$

   Bring all the $\frac{dy}{dx}$ terms together:

   $$\left(\frac{1}{3}y^{-2/3} - 4x\right)\frac{dy}{dx} = 4y - \frac{1}{3}x^{-2/3} \Rightarrow \frac{dy}{dx} = \frac{4y - \frac{1}{3}x^{-2/3}}{\frac{1}{3}y^{-2/3} - 4x}$$

(d) $\sqrt{x+y} = \sqrt[3]{x-y}$

Another implicit differentation:

$$\frac{1}{2}(x+y)^{-1/2}(1+y') = \frac{1}{3}(x-y)^{-2/3}(1-y')$$

Multiply out so that we can isolate $y'$

$$\frac{1}{2}(x+y)^{-1/2} + y' \cdot \frac{1}{2}(x+y)^{-1/2} = \frac{1}{3}(x-y)^{-2/3} - y' \cdot \frac{1}{3}(x-y)^{-2/3}$$

Now isolate $y'$

$$y'\left(\frac{1}{2}(x+y)^{-1/2} + \frac{1}{3}(x-y)^{-2/3}\right) = \frac{1}{3}(x-y)^{-2/3} - \frac{1}{2}(x+y)^{-1/2}$$

Final answer:

$$y' = \frac{\frac{1}{3}(x-y)^{-2/3} - \frac{1}{2}(x+y)^{-1/2}}{\frac{1}{2}(x+y)^{-1/2} + \frac{1}{3}(x-y)^{-2/3}}$$

(e) $y = \sin(2\cos(3x))$

Chain Rule:

$$y' = \cos(2\cos(3x)) \cdot (-2\sin(3x)) \cdot 3 = -6\cos(2\cos(3x))\sin(3x)$$

(f) $y = (\cos(x))^{2x}$

Logarithmic Differentiation:

$$\ln(y) = 2x \cdot \ln(\cos(x)) \Rightarrow \frac{1}{y}y' = 2\ln(\cos(x)) + 2x\frac{1}{\cos(x)} \cdot \sin(x) = 2\ln(\cos(x)) + 2x\tan(x)$$

so that

$$y' = (\cos(x))^{2x}(2\ln(\cos(x)) + 2x\tan(x))$$

(g) $y = (\tan^{-1}(x))^{-1}$

Chain Rule:

$$y' = -(\tan^{-1}(x))^{-2} \cdot \frac{1}{x^2+1}$$

(h) $y = \sin^{-1}(\cos^{-1}(x))$

Chain Rule:

$$y' = \frac{1}{\sqrt{1-(\cos^{-1}(x))^2}} \cdot \frac{-1}{\sqrt{1-x^2}}$$

(i) $y = \log_{10}(x^2 - x)$

First, we don't know the derivative of $\log_{10}(x)$, so we can re-write the problem as:

$$10^y = x^2 - x$$

Now differentiate implicitly:

$$10^y \ln(10) \cdot y' = 2x - 1$$

Now, $10^y = x^2 - x$, so:

$$(x^2 - x)\ln(10) \cdot y' = 2x - 1$$

and

$$y' = \frac{2x - 1}{(x^2 - x)\ln(10)}$$

Alternatively, we can compute the derivative of $\log_{10}(x)$: If $y = \log_{10}(x)$, then $10^y = x$, and $10^y \ln(10) \cdot y' = 1$. Now, $10^y = x$, so

$$x \ln(10) \cdot y' = 1 \Rightarrow y' = \frac{1}{x \ln(10)}$$

and now we can use the chain rule:

$$y' = \frac{1}{(x^2 - x)\ln(10)} \cdot 2x - 1$$

(j) $y = x^{x^2+2}$

Logarithmic Differentiation:

$$\ln(y) = (x^2 + 2)\ln(x)$$

Use the product rule:

$$\frac{1}{y}y' = (2x)\ln(x) + (x^2 + 2)\frac{1}{x} = 2x\ln(x) + x + \frac{2}{x}$$

Final answer:

$$y' = x^{x^2+2}\left(2x\ln(x) + x + \frac{2}{x}\right)$$

(k) $y = \mathrm{e}^{\cos(x)} + \sin(5^x)$

Chain rule:

$$y' = \mathrm{e}^{\cos(x)}(-\sin(x)) + \cos(5^x) \cdot 5^x \ln(5)$$

(l) $y = \cot(3x^2 + 5)$

Chain rule:

$$y' = -\csc^2(3x^2 + 5) \cdot (6x) = -6x\csc^2(3x^2 + 5)$$

(m) $y = \sqrt{\sin(\sqrt{x})}$

Chain rule:

$$y' = \frac{1}{2}\left(\sin(\sqrt{x})\right)^{-1/2} \cdot \cos(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2}$$

(n) $\sqrt{x} + \sqrt[3]{y} = 1$

Implicit Differentiation:

$$\frac{1}{2}x^{-1/2} + \frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0$$

so

$$\frac{dy}{dx} = -\frac{3y^{2/3}}{2x^{1/2}}$$

(o) $x\tan(y) = y - 1$

Product rule/Implicit Diff

$$\tan(y) + x\sec^2(y)y' = y' \Rightarrow \tan(y) = y'(1 - x\sec^2(y))$$

Solve for $y'$:

$$y' = \frac{\tan(y)}{1 - x\sec^2(y)}$$

(p) $y = \frac{-2}{\sqrt[4]{t^3}}$, where $t = \ln(x^2)$.

First, note that $y = -2t^{-3/4}$ and $t = 2\ln(x)$. Now,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

where

$$\frac{dy}{dt} = \frac{-3}{4}t^{-7/4}, \qquad \frac{dt}{dx} = \frac{2}{x}$$

so put it all together (and substitute back for $x$):

$$\frac{dy}{dx} = \frac{-3}{4}\left(2\ln(x)\right)^{-7/4} \cdot \frac{2}{x} = \frac{-6}{4x(2\ln(x))^{7/4}}$$

(q) $y = x3^{-1/x}$

$$y' = 3^{-1/x} + x3^{-1/x} \cdot \frac{1}{x^2} = 3^{-1/x}\left(1 + \frac{1}{x}\right)$$

8. Let $f(x) = x2^{x+1}$. Without explicitly computing the inverse, what is the equation of the tangent line to $f^{-1}(x)$ at $x = 4$? HINT: The point $(1, 4)$ goes through the graph of $f$.

The general rule here is the following: If $(a, b)$ is on the graph of $f$, then $(b, a)$ is on the graph of $f^{-1}$. We also said that the derivative of $f^{-1}$ at $b$ is $\frac{1}{f'(a)}$, given that $f'(a) \neq 0$. So, in this problem,

$$f'(x) = 2^{x+1} + x2^{x+1}\ln(2) \cdot 1$$

and $f'(1) = 4 + 4\ln(2)$, so the derivative of $f^{-1}$ at 4 is $\frac{1}{4+4\ln(2)}$. Now we have a slope and a point:

$$y - 1 = \frac{1}{4 + 4\ln(2)}(x - 4)$$

is the tangent line to $f^{-1}$ at $x = 4$.

9. Find the local maximums and minimums: $f(x) = x^3 - 3x + 1$ Show your answer is correct by using both the first derivative test and the second derivative test.

   To find local maxs and mins, first differentiate to find critical points:

$$f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$$

   For the first derivative test, set up a sign chart. You should see that $3x^2 - 3 = 3(x+1)(x-1)$ is positive for $x < 1$ and $x > 1$, and $f'(x)$ is negative if $-1 < x < 1$. Therefore, at $x = -1$, the derivative changes sign from positive to negative, so $x = -1$ is the location of a local maximum. At $x = 1$, the derivative changes sign from negative to positive, so we have a local minimum.

   For the second derivative test, we compute the second derivative at the critical points:

$$f''(x) = 6x$$

   so at $x = -1$, $f$ is concave down, so we have a local max, and at $x = 1$, $f$ is concave up, so we have a local min.

10. Compute the limit, if it exists. You may use any method (except a numerical table).

   (a) $\displaystyle\lim_{x\to 0} \frac{x - \sin(x)}{x^3}$
   We have a form of $\frac{0}{0}$, so use L'Hospital's rule:

$$\lim_{x\to 0} \frac{x - \sin(x)}{x^3} = \lim_{x\to 0} \frac{1 - \cos(x)}{3x^2}$$

   We still have $\frac{0}{0}$, so do it again and again!

$$\lim_{x\to 0} \frac{1 - \cos(x)}{3x^2} = \lim_{x\to 0} \frac{\sin(x)}{6x} = \lim_{x\to 0} \frac{\cos(x)}{6} = \frac{1}{6}$$

   (b) $\displaystyle\lim_{x\to 0} \frac{1 - e^{-2x}}{\sec(x)}$
   Note that $\sec(0) = \frac{1}{\cos(0)} = 1$, so this function is continuous at $x = 0$ (we can substitute $x = 0$ in directly), and we get that the limit is 0.

   (c) $\displaystyle\lim_{x\to 4^+} \frac{x - 4}{|x - 4|}$
   Rewrite the expression to get rid of the absolute value:

$$\frac{x-4}{|x-4|} = \begin{cases} \frac{x-4}{x-4} & \text{if } x > 4 \\ \frac{x-4}{-(x-4)}, & \text{if } x < 4 \end{cases} = \begin{cases} 1 & \text{if } x > 4 \\ -1 & \text{if } x < 4 \end{cases}$$

   Therefore,

$$\lim_{x\to 4^+} \frac{x-4}{|x-4|} = 1$$

   (Note that the overall limit does not exist, however).

(d) $\lim_{x\to-\infty}\sqrt{\frac{2x^2-1}{x+8x^2}}$

For this problem, we should recall that if $x<0$, then $x=-\sqrt{x^2}$, although in this particular case, the negative signs will cancel:

$$\lim_{x\to-\infty}\sqrt{\frac{2x^2-1}{x+8x^2}\cdot\frac{\frac{-1}{\sqrt{x^2}}}{\frac{-1}{\sqrt{x^2}}}}=\lim_{x\to-\infty}\sqrt{\frac{2-\frac{1}{x^2}}{\frac{1}{x}+8}}=\sqrt{\frac{2}{8}}=\frac{1}{2}$$

(e) $\lim_{x\to\infty}\sqrt{x^2+x+1}-\sqrt{x^2-x}$

Multiply by the conjugate (or rationalize):

$$\lim_{x\to\infty}\sqrt{x^2+x+1}-\sqrt{x^2-x}\cdot\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}=\lim_{x\to\infty}\frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}.$$

Now divide numerator and denominator:

$$\lim_{x\to\infty}\frac{2x+1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}\cdot\frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2}}}=\lim_{x\to\infty}\frac{2+\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}}}=1$$

(f) $\lim_{h\to 0}\frac{(1+h)^{-2}-1}{h}$

For practice, we'll try it without using L'Hospital's rule:

$$\lim_{h\to 0}\frac{(1+h)^{-2}-1}{h}=\lim_{h\to 0}\frac{\frac{1}{(1+h)^2}-1}{h}=\lim_{h\to 0}\frac{1-(1+h)^2}{h(1+h)^2}=\lim_{h\to 0}\frac{-2h-h^2}{h(1+h)^2}=-2$$

With L'Hospital:

$$\lim_{h\to 0}\frac{(1+h)^{-2}-1}{h}=\lim_{h\to 0}\frac{-2(1+h)^{-3}}{1}=-2$$

EXTRA: This limit was the derivative of some function at some value of $x$. Name the function and the $x$ value[1].

(g) $\lim_{x\to\infty}x^3e^{-x^2}$

First rewrite the function so that it's in an acceptable form for L'Hospital's rule:

$$\lim_{x\to\infty}\frac{x^3}{e^{x^2}}=\lim_{x\to\infty}\frac{3x^2}{2xe^{x^2}}$$

Note that $\frac{3x^2}{2xe^{x^2}}=\frac{3x}{2e^{x^2}}$, and again use L'Hospital's rule:

$$\lim_{x\to\infty}\frac{3x}{2e^{x^2}}=\lim_{x\to\infty}\frac{3}{4xe^{x^2}}=0$$

[1]The function is $f(x)=x^{-2}$ at $x=1$

(h) $\lim_{x\to 1} \dfrac{x^{1000}-1}{x-1}$
Using L'Hospital's rule:

$$\lim_{x\to 1} \frac{x^{1000}-1}{x-1} = \lim_{x\to 1} \frac{1000x^{999}}{1} = 1000$$

(i) $\lim_{x\to 0} \dfrac{x}{\tan^{-1}(4x)}$
Recall that $\tan^{-1}(0) = 0$, since $\tan(0) = 0$, so this is in a form for L'Hospital's rule:

$$\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)} = \lim_{x\to 0} \frac{1}{\frac{1}{1+(4x)^2}\cdot 4} = \lim_{x\to 0} \frac{1+16x^2}{4} = \frac{1}{4}$$

(j) $\lim_{x\to 1} x^{\frac{1}{1-x}}$
In this case, recall that $x^{\frac{1}{1-x}} = \mathrm{e}^{\frac{1}{1-x}\cdot \ln(x)}$, so:

$$\lim_{x\to 1} x^{\frac{1}{1-x}} = \lim_{x\to 1} \mathrm{e}^{\frac{1}{1-x}\cdot \ln(x)} = \mathrm{e}^{\lim_{x\to 1} \frac{\ln(x)}{1-x}}$$

so we focus on the exponent:

$$\lim_{x\to 1} \frac{\ln(x)}{1-x} = \lim_{x\to 1} \frac{\frac{1}{x}}{-1} = -1$$

so the overall limit:

$$\lim_{x\to 1} x^{\frac{1}{1-x}} = \mathrm{e}^{-1}$$

11. Determine all vertical/horizontal asymptotes and critical points of $f(x) = \frac{2x^2}{x^2-x-2}$
The vertical asymptotes: $x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$, so $x = -1, x = 2$ are the equations of the vertical asymptotes (note that the numerator is not zero at these values).

The horizontal asymptotes:

$$\lim_{x\to \pm\infty} \frac{2x^2}{x^2-x-2} = \lim_{x\to \pm\infty} \frac{2}{1-\frac{1}{x}-\frac{2}{x^2}} = 2$$

so $y = 2$ is the vertical asymptote (for both $+\infty$ and $-\infty$).

12. Find values of $m$ and $b$ so that (1) $f$ is continuous, and (2) $f$ is differentiable.

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \le 2 \\ mx+b & \text{if} \quad x > 2 \end{cases}$$

First we see that if $x < 2$, $f(x) = x^2$ which is continuous, and if $x < 2$, $f(x) = mx + b$, which is also continuous for any value of $m$ and $b$. The only problem point is $x = 2$, so we check the three conditions from the definition of continuity:

- $f(2) = 2m + b$, so $f(2)$ exists.
- To compute the limit, we have to do them separately:

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} x^2 = 4$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} mx + b = 2m + b$$

  For the limit to exist, we must have $4 = 2m + b$. This will also automatically make item 3 true.

  There are an infinite number of possible solutions. Given any $m$, $b = 4 - 2m$.

For the second part, we know that $f$ must be continuous to be differentiable, so that leaves us with $b = 4 - 2m$. Also, the derivatives need to match up at $x = 2$. On the right side of $x = 2$, $f'(x) = 2x$ and on the left side of $x = 2$, $f'(x) = m$. Therefore, $4 = m$ and $b = 4 - 2 \cdot 4 = -4$.

To be differentiable at $x = 2$, we require $m = 4$ and $b = -4$.

13. Find the local and global extreme values of $f(x) = \frac{x}{x^2+x+1}$ on the interval $[-2, 0]$.

We see that $x^2 + x + 1 = 0$ has no solution, so $f(x)$ is continuous on $[-2, 0]$. Therefore, the extreme value theorem is valid. Next, find the critical points:

$$f'(x) = \frac{(x^2 + x + 1) - x(2x + 1)}{(x^2 + x + 1)^2} = \frac{-x^2 + 1}{(x^2 + x + 1)^2}$$

so the critical points are $x = \pm 1$ of which we are only concerned with $x = -1$. Now build a chart of values:

| $x$ | $0$ | $-1$ | $-2$ |
|---|---|---|---|
| $f(x)$ | $0$ | $-1$ | $-2/3$ |

The minimum occurs at $x = -1$ and the maximum occurs at $x = 0$. The minimum value is $-1$ and the maximum value is $0$.

For the local min/max, use a sign chart for $f'(x)$ in the interval $[-2, 0]$ (or you can use the second derivative test). We see that the denominator of $f'(x)$ is always positive, and the numerator is $1 - x^2$, which changes from negative to positive at $x = -1$, so that $f$ is decreasing then increasing and there is a local minimum at $x = -1$.

14. Suppose $f$ is differentiable so that:

$$f(1) = 1,\ f(2) = 2,\ f'(1) = 1\ f'(2) = 2$$

If $g(x) = f(x^3 + f(x^2))$, evaluate $g'(1)$.

Use the chain rule to get that:

$$g'(x) = f'\left(x^3 + f(x^2)\right) \cdot \left(3x^2 + f'(x^2) \cdot 2x\right)$$

Be careful with the parentheses!:

$$g'(1) = f'(1 + f(1)) \cdot (3 + 2f'(1)) = f'(1 + 1)(3 + 2) = 5f'(2) = 5 \cdot 2 = 10$$

15. Let $x^2y + a^2xy + \lambda y^2 = 0$

(a) Let $a$ and $\lambda$ be constants, and let $y$ be a function of $x$. Calculate $\frac{dy}{dx}$:

$$2xy + x^2\frac{dy}{dx} + a^2y + a^2x\frac{dy}{dx} + 2\lambda y\frac{dy}{dx} = 0$$

$$(x^2 + a^2x + 2\lambda y)\frac{dy}{dx} = -(2xy + a^2y) \Rightarrow \frac{dy}{dx} = \frac{-(2xy + a^2y)}{x^2 + a^2x + 2\lambda y}$$

(b) Let $x$ and $y$ be constants, and let $a$ be a function of $\lambda$. Calculate $\frac{da}{d\lambda}$:

$$2axy\frac{da}{d\lambda} + y^2 = 0 \Rightarrow \frac{da}{d\lambda} = \frac{-y^2}{2axy}$$

EXTRA[2]: What is $\frac{d\lambda}{da}$?

16. Show that $x^4 + 4x + c = 0$ has at most one solution in the interval $[-1, 1]$.

We don't need the Intermediate Value Theorem here, only the Mean Value Theorem. The derivative is $4x^3 + 4$, so the only critical point is $x = -1$, which is also an endpoint. This implies: (1) If $x^4 + 4x + c = 0$ had two solutions (which is possible), then one of them must be outside the interval, since the two solutions must be on either side of $x = -1$. Therefore, there could be one solution inside the interval. (2) There cannot be any other solution to $x^4 + 4x + c = 0$ inside the interval, because then there would have to be another critical point in $[-1, 1]$. Therefore, we conclude that there is at most one solution inside the interval (there might be no solutions).

17. True or False, and give a short explanation.

(a) If $f$ has an absolute minimum at $c$, then $f'(c) = 0$.
False. For example, $f(x) = |x|$ has an absolute minimum at $x = 0$, but $f'(x)$ is not defined at $x = 0$.

(b) If $f$ is differentiable, then

$$\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$$

True, since

$$\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2}(f(x))^{-1/2} \cdot f'(x) = \frac{f'(x)}{2\sqrt{f(x)}}$$

(c) $\frac{d}{dx}(10^x) = x10^{x-1}$
False. We cannot use the Power Rule, since there is a variable in the exponent. The correct derivative is found using the rule for $a^x$:

$$\frac{d}{dx}(10^x) = 10^x \ln(10)$$

[2]The answer is $\frac{d\lambda}{da} = \frac{2axy}{-y^2}$

(d) If $f'(x)$ exists and is nonzero for all $x$, then $f(1) \neq f(0)$.

True. If $f'(x)$ exists for all $x$, then $f$ is differentiable everywhere (and is also continuous everywhere). Thus, the Mean Value Theorem applies. If $f(1) = f(0)$, that would imply the existence of a $c$ in the interval $(0, 1)$ so that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = 0$$

but we're told that $f'(x) \neq 0$.

(e) If $y = ax + b$, then $\frac{dy}{da} = x$

True. If we're computing $\frac{dy}{da}$, then we're treating $x$ and $b$ as constants. Differentiating, we get

$$\frac{dy}{da} = x + 0 = x$$

(f) If $2x + 1 \leq f(x) \leq x^2 + 2$ for all $x$, then $\lim_{x\to 1} f(x) = 3$.

True. This is the Squeeze Theorem. If $f(x)$ is trapped between $2x + 1$ and $x^2 + 2$ for all $x$, and since the limit as $x \to 1$ of $2x + 1$ is 3, and the limit as $x \to 1$ of $x^2 + 2 = 3$, then that forces the limit as $x \to 1$ of $f(x)$ to also be 3.

(g) If $f'(r)$ exists, then

$$\lim_{x\to r} f(x) = f(r)$$

True. The statement that $f'(r)$ exists says that $f$ is differentiable at $r$. The statement that $\lim_{x\to r} f(x) = f(r)$ is asking if $f$ is continuous at $r$. We know that all differentiable functions are continuous, so the statement is True.

(h) If $f$ and $g$ are differentiable, then:

$$\frac{d}{\mathrm{dx}}(f(g(x)) = f'(x)g'(x)$$

False. The chain rule states that $\frac{d}{\mathrm{dx}}(f(g(x)) = f'(g(x))g'(x)$

(i) If $f(x) = x^2$, then the equation of the tangent line at $x = 3$ is: $y - 9 = 2x(x - 3)$

False. $2x$ is a formula for the slope, not the slope itself. The equation of the tangent line is: $y - 9 = 6(x - 3)$.

(j) $\lim_{\theta\to\frac{\pi}{3}} \dfrac{\cos(\theta) - \frac{1}{2}}{\theta - \frac{\pi}{3}} = -\sin\left(\dfrac{\pi}{3}\right)$

True by l'Hospital's rule. You could have also said that the expression is in the form:

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

with $f(x) = \cos(x)$ and $a = \frac{\pi}{3}$. This is another way of defining the derivative of $\cos(x)$ at $x = \frac{\pi}{3}$.

(k) There is no solution to $e^x = 0$

True. If there were a solution, it would be $x = \ln(0)$, but $\ln(0)$ is not defined.

(l) $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$
False, with the usual restrictions on the sine function. That is, if $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then it is true that $\sin^{-1}(\sin(\theta)) = \theta$. In this case,

$$\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$$

(m) $5^{\log_5(2x)} = 2x$, for $x > 0$.
True, since $5^x$ and $\log_5(x)$ are inverses of each other. We needed $x > 0$ so that $\log_5(2x)$ is defined.

(n) $\frac{d}{dx}\ln(|x|) = \frac{1}{x}$, for all $x \neq 0$.
True:

$$\ln|x| = \begin{cases} \ln(x), & \text{if } x > 0 \\ \ln(-x), & \text{if } x < 0 \end{cases}$$

so

$$\frac{d}{dx}\ln|x| = \begin{cases} \frac{1}{x}, & \text{if } x > 0 \\ \frac{1}{-x}\cdot(-1) = \frac{1}{x}, & \text{if } x < 0 \end{cases}$$

(o) $\frac{d}{dx}10^x = x10^{x-1}$
Oops- a duplication. See part (c)

(p) If $x > 0$, then $(\ln(x))^6 = 6\ln(x)$
False. The rule says: $\log(a^b) = b\log(a)$, but here the 6 is outside the logarithm.

(q) False. The most general antiderivative is a piecewise defined function, since $x = 0$ is not in the domain of $f$:

$$f(x) = \begin{cases} \frac{-1}{x} + C_1, & \text{if} \quad x < 0 \\ \frac{-1}{x} + C_2, & \text{if} \quad x > 0 \end{cases}$$

18. Find the domain of $\ln(x - x^2)$:

Use a sign chart to determine where $x - x^2 = x(1 - x) > 0$:

| | | | |
|---|---|---|---|
| $x$ | $-$ | $+$ | $+$ |
| $1-x$ | $+$ | $+$ | $-$ |
| | $x < 0$ | $0 < x < 1$ | $x > 1$ |

so overall, $x - x^2 > 0$ if $0 < x < 1$.

19. Find the value of $c$ guaranteed by the Mean Value Theorem, if $f(x) = \frac{x}{x+2}$ on the interval $[1, 4]$.

To set things up, we see that $f$ is continuous on $[1, 4]$ and differentiable on $(1, 4)$, since the only "bad point" is $x = -2$. We should get that $f'(x) = \frac{2}{(x+2)^2}$, $f(1) = \frac{1}{3}$ and $f(4) = \frac{2}{3}$. Therefore, the Mean Value Theorem says that $c$ should satisfy:

$$\frac{2}{(c+2)^2} = \frac{\frac{2}{3} - \frac{1}{3}}{4 - 1} = \frac{1}{9}$$

or

$$(c+2)^2 = 18 \Rightarrow c = -2 \pm \sqrt{18}$$

of which only $-2+\sqrt{18} \approx 2.243$ is inside our interval.

20. Given that the graph of $f$ passes through the point $(1,6)$ and the slope of the tangent line at $(x, f(x))$ is $2x+1$, find $f(2)$.

    Since $f'(x) = 2x+1$, $f(x) = x^2+x+C$ is the general antiderivative. Given that $(1,6)$ goes through $f$, $1^2+1+C = 6 \Rightarrow C = 4$. Therefore, $f(x) = x^2+x+4$. Now, $f(2) = 4+2+4 = 10$.

21. A fly is crawling from left to right along the curve $y = 8-x^2$, and a spider is sitting at $(4,0)$. At what point along the curve does the spider first see the fly?

    Another way to say this: What are the tangent lines through $y = 8-x^2$ that also go through $(4,0)$?

    The unknown value here is the $x-$coordinate, so let $x = a$. Then the slope is $-2a$, and the corresponding point on the curve is $(a, 8-a^2)$. The general form of the equation of the tangent line is then given by:

$$y - 8 + a^2 = -2a(x-a)$$

    where $x, y$ are points on the tangent line. We want the tangent line to go through $(4,0)$, so we put this point in and solve for $a$:

$$-8+a^2 = -2a(4-a) = -8a+2a^2 \quad \Rightarrow \quad 0 = a^2-8a+8$$

$$\Rightarrow \quad a = \frac{8\pm\sqrt{32}}{2}$$

    so we take the leftmost value, $a = \frac{8-\sqrt{32}}{2}$.

22. Compute the limit, without using L'Hospital's Rule. $\displaystyle\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7}$

    Rationalize to get:

$$\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7} \cdot \frac{\sqrt{x+2}+3}{\sqrt{x+2}+3} = \lim_{x\to 7} \frac{x+2-9}{(x-7)(\sqrt{x+2}+3)}$$

$$= \lim_{x\to 7} \frac{1}{(\sqrt{x+2}+3)} = \frac{1}{6}$$

    which is the derivative of $\sqrt{x+2}$ at $x = 7$.

23. For what value(s) of $c$ does $f(x) = cx^4 - 2x^2 + 1$ have both a local maximum and a local minimum?

    First, $f'(x) = 4cx^3 - 4x = 4x(cx^2-1)$, and $f''(x) = 12cx^2 - 4$. The candidates for the location of the local max's and min's are where $f'(x) = 0$, which are $x = 0$ and $x = \pm\sqrt{1/c}$ $(c > 0)$. We can use the second derivative test to check these out:

At $x = 0$, $f''(0) = -4$, so $x = 0$ is always a local max. At $x = \pm\sqrt{1/c}$, $f''(\pm\sqrt{1/c}) = 12 - 4 = 8$. So, if $c > 0$, there are local mins at $x = \pm\sqrt{1/c}$.

24. If $f(x) = \sqrt{1 - 2x}$, determine $f'(x)$ *by using the definition of the derivative.*

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{1-2(x+h)} - \sqrt{1-2x}}{h} =$$

$$\lim_{h\to 0} \frac{\sqrt{1-2(x+h)} - \sqrt{1-2x}}{h} \cdot \frac{\sqrt{1-2(x+h)} + \sqrt{1-2x}}{\sqrt{1-2(x+h)} + \sqrt{1-2x}} =$$

$$\lim_{h\to 0} \frac{1 - 2x - 2h - 1 + 2x}{h(\sqrt{1-2(x+h)} + \sqrt{1-2x})} =$$

$$\lim_{h\to 0} \frac{-2}{\sqrt{1-2(x+h)} + \sqrt{1-2x}} = \frac{-1}{\sqrt{1-2x}}$$

25. A *point of inflection* for a function $f$ is the $x$ value for which $f''(x)$ changes sign (either from positive to negative or vice versa).

> If $f''$ is continuous, and $f''(a) < 0$ and $f''(b) > 0$, there is a $c$ so that $f''(c) = 0$ is an inflection point.

...which is the Intermediate Value Theorem.

Find constants $a$ and $b$ so that $(1, 6)$ is an inflection point for $y = x^3 + ax^2 + bx + 1$.

Differentiate twice to get:

$$y'' = 6x + 2a$$

At $x = 1$, we want an inflection point, so $6 + 2a$ should be a point where $y''$ changes sign: $6 + 2a = 0 \Rightarrow a = -3$. We see that if $a < -3$, then $y'' < 0$, and if $a > -3$, $y'' > 0$.

Putting this back into the function, we have:

$$y = x^3 - 3x^2 + bx + 1$$

and we want the curve to go through the point $(1, 6)$:

$$6 = 1 - 3 + b + 1$$

so $b = 7$.

26. Suppose that $F(x) = f(g(x))$ and $g(3) = 6$, $g'(3) = 4$, $f(3) = 2$ and $f'(6) = 7$. Find $F'(3)$.

By the Chain Rule:

$$F'(3) = f'(g(3))g'(3)$$

so $F'(3) = f'(6) \cdot 4 = 7 \cdot 4 = 28$

27. Find the dimensions of the rectangle of largest area that has its base on the $x-$axis and the other two vertices on the parabola $y = 8 - x^2$.

Try drawing a picture first: The parabola opens down, goes through the $y-$intercept at 8, and has $x-$intercepts of $\pm\sqrt{8}$.

Now, let $x$ be as usual, so that the full length of the base of the rectangle is $2x$. Then the height is $y$, or $8 - x^2$. Therefore, the area of the rectangle is:

$$A = 2xy = 2x(8 - x^2) = 16x - 2x^3$$

and $0 \le x \le \sqrt{8}$. We see that the area will be zero at the endpoints, so we expect a maximum at the critical point inside the interval:

$$\frac{dA}{dx} = 16 - 6x^2$$

so the critical points are $x = \pm\frac{4}{\sqrt{6}}$, of which only $x = \frac{4}{\sqrt{6}}$ is in our interval. So the dimensions of the rectangle are:

$$2x = \frac{8}{\sqrt{6}}, y = 5\frac{1}{3}$$

28. Let $G(x) = h(\sqrt{x})$. Then where is $G$ differentiable? Find $G'(x)$.

First compute $G'(x) = h'(\sqrt{x})\frac{1}{2}x^{-1/2}$. From this we see that as long as $h$ is differentiable and $x > 0$, then $G$ will be differentiable.

29. If position is given by: $f(t) = t^4 - 2t^3 + 2$, find the times when the acceleration is zero. Then compute the velocity at these times.

Take the second derivative, and set it equal to zero:

$$f'(x) = 4t^3 - 6t^2, \quad f''(t) = 12t^2 - 12t = 0 \Rightarrow t = 0, t = 1$$

The velocity at $t = 0$ is 0 and the velocity at $t = 1$ is $4 - 6 = -2$.

30. If $y = \sqrt{5t - 1}$, compute $y'''$.

Nothing tricky here- Just differentiate, and differentiate, and differentiate!

$$y' = \frac{1}{2}(5t-1)^{-1/2} \cdot 5 = \frac{5}{2}(5t-1)^{-1/2}$$

$$y'' = \frac{5}{2} \cdot \frac{-1}{2}(5t-1)^{-3/2} \cdot 5 = \frac{-25}{4}(5t-1)^{-3/2}$$

$$y''' = \frac{375}{8}(5t-1)^{-5/2}$$

31. Find a second degree polynomial so that $P(2) = 5$, $P'(2) = 3$, and $P''(2) = 2$.

The general form of a second degree polynomial is $ax^2 + bx + c$, so we need to find $a, b, c$. So let $P(x) = ax^2 + bx + c$. Then $P'(x) = 2ax + b$, and $P''(x) = 2a$.

$P''(2) = 2$, so $2a = 2$ and $a = 1$. Now, $P'(x) = 2x + b$, and $P'(2) = 3$, so $3 = 4 + b$, and $b = -1$. Finally, $P(x) = x^2 - x + c$ and $P(2) = 5$, so $5 = 4 - 2 + c$, so $c = 3$.

$$P(x) = x^2 - x + 3$$

32. Find a function $f(x)$ so that $f'(x) = 4 - 3(1 + x^2)^{-1}$, and $f(1) = 0$

Antidifferentiate:

$$f(x) = 4x - 3\tan^{-1}(x) + C$$

and $f(1) = 0$ means:

$$4 - 3\tan^{-1}(1) + C = 0$$

Now, $\tan^{-1}(1) = \frac{\pi}{4}$ (it's where $\sin(x) = \cos(x)$), so $C = \frac{3\pi}{4} - 4$. so $f(x) = 4x - 3\tan^{-1}(x) + \frac{3\pi}{4} - 4$

33. If $f(x) = (2 - 3x)^{-1/2}$, find $f(0)$, $f'(0)$, $f''(0)$.

Differentiate:

$$f'(x) = \frac{-1}{2}(2 - 3x)^{-3/2}(-3) = \frac{3}{2}(2 - 3x)^{-3/2}$$
$$f''(x) = \frac{-9}{4}(2 - 3x)^{-5/2}(-3) = \frac{27}{4}(2 - 3x)^{-5/2}$$

Now, (note that $2^{3/2} = 2\sqrt{2}$ and $2^{5/2} = 4\sqrt{2}$):

$$f(0) = \frac{1}{\sqrt{2}}, \quad f'(0) = \frac{3}{2} \cdot \frac{1}{2^{3/2}} = \frac{3}{4\sqrt{2}}, \quad f''(0) = \frac{27}{16\sqrt{2}}$$

34. Car A is traveling west at 50 mi/h, and car B is traveling north at 60 mi/h. Both are headed for the intersection between the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

Let $A(t)$, $B(t)$ be the positions of cars $A$ and $B$ at time $t$. Let the distance between them be $z(t)$, so that the Pythagorean Theorem gives:

$$z^2 = A^2 + B^2$$

Translating the question, we get that we want to find $\frac{dz}{dt}$ when $A = 0.3$, $B = 0.4$, (so $z = 0.5$), $A'(t) = 50$, $B'(t) = 60$. Then:

$$2z\frac{dz}{dt} = 2A\frac{dA}{dt} + 2B\frac{dB}{dt}$$

The two's divide out and put in the numbers:

$$0.5 \cdot \frac{dz}{dt} = 0.3 \cdot 50 + 0.4 \cdot 60$$

and solve for $\frac{dz}{dt}$, 78.

35. Compute $\Delta y$ and $dy$ for the value of $x$ and $\Delta x$: $f(x) = 6 - x^2$, $x = -2$, $\Delta x = 0.4$.

Recall that $\Delta y = f(x+\Delta x) - f(x)$ and $\Delta x = dx$. Also, $dy = f'(x)\,dx$. Put the numbers in:

$$\Delta y = f(-2 + 0.4) - f(-2) = 3.44 - 2 = 1.44$$
$$dy = -2x\;dx = -2 \cdot 2 \cdot (0.4) = 1.6$$

36. Find the linearization of $f(x) = \sqrt{1-x}$ at $x = 0$.

To linearize, we find the equation of the tangent line.

$$f'(x) = \frac{1}{2}(1-x)^{-1/2}(-1)$$

so $f'(0) = -\frac{1}{2}$, and the point is $(0, 1)$.

$$y - 1 = -\frac{1}{2}x, \quad \text{or} \quad y = -\frac{1}{2} + 1$$

37. Find $f(t)$, if $f''(t) = t + \sqrt{t}$, and $f(1) = 1$, $f'(1) = 2$.

$$f'(t) = \frac{1}{2}t^2 + \frac{2}{3}t^{3/2} + C$$

so $f'(1) = 2$ means:

$$\frac{1}{2} + \frac{2}{3} + C = 2, \text{ so } C = \frac{5}{6}$$

Now, $f'(t) = \frac{1}{2}t^2 + \frac{2}{3}t^{3/2} + \frac{5}{6}$, and

$$f(t) = \frac{1}{2} \cdot \frac{1}{3}t^3 + \frac{2}{3} \cdot \frac{2}{5}t^{5/2} + \frac{5}{6}t + C = \frac{1}{6}t^3 + \frac{4}{15}t^{5/2} + \frac{5}{6}t + C$$

Now, $f(1) = 1$ means:

$$\frac{1}{6} + \frac{4}{15} + \frac{5}{6} + C = 1 \Rightarrow \frac{5+8+25}{30} + C = 1 \Rightarrow C = 1 - \frac{19}{15} = \frac{-4}{15}$$

38. Find $f'(x)$ directly from the definition of the derivative (using limits and without L'Hospital's rule):

First, recall that:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We'll use this for these exercises:

(a) $f(x) = \sqrt{3 - 5x}$

$$\lim_{h \to 0} \frac{\sqrt{3 - 5x - 5h} - \sqrt{3 - 5x}}{h} \cdot \frac{\sqrt{3 - 5x - 5h} + \sqrt{3 - 5x}}{\sqrt{3 - 5x - 5h} + \sqrt{3 - 5x}}$$

$$\lim_{h \to 0} \frac{3 - 5x - 5h - 3 + 5x}{h\left(\sqrt{3 - 5x - 5h} + \sqrt{3 - 5x}\right)}$$

$$\frac{-5}{2\sqrt{3 - 5x}}$$

(b) $f(x) = x^2$

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} =$$

$$\lim_{h\to 0} \frac{h(2x+h)}{h} = 2x$$

(c) $f(x) = x^{-1}$

$$\lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} =$$

$$\lim_{h\to 0} \frac{-h}{hx(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$$

39. If $f(0) = 0$, and $f'(0) = 2$, find the derivative of $f(f(f(f(x))))$ at $x = 0$.

    First, note that the derivative is (Chain Rule):

$$f'(f(f(f(x)))) \cdot f'(f(f(x))) \cdot f'(f(x)) \cdot f'(x)$$

which simplifies (since $f(0) = 0$) to:

$$f'(0) \cdot f'(0) \cdot f'(0) \cdot f'(0) = 2^4 = 16$$

40. Differentiate:

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{x} & \text{if } x < 0 \end{cases}$$

Is $f$ differentiable at $x = 0$? Explain.

**Correction:** $-\sqrt{x} = -\sqrt{-x}$

Is $f$ differentiable at $x = 0$? Explain.

$f$ will not be differentiable at $x = 0$. Note that, if $x > 0$, then $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(x) \to \infty$ as $x \to 0^+$

If $x < 0$, $f'(x) = \frac{1}{2\sqrt{-x}}$, which also goes to infinity as $x$ approaches 0 (from the left).

41. $f(x) = |\ln(x)|$. Find $f'(x)$.

    We can rewrite $f$ (Recall that $\ln(x) < 0$ if $0 < x < 1$)

$$f(x) = \begin{cases} \ln(x), & \text{if } x \geq 1 \\ -\ln(x), & \text{if } 0 < x < 1 \end{cases}$$

and differentiate piecewise:

$$f'(x) = \begin{cases} \frac{1}{x}, & \text{if } x > 1 \\ -\frac{1}{x}, & \text{if } 0 < x < 1 \end{cases}$$

Note that the pieces don't match at $x = 1$; we remove that point from the domain.

42. $f(x) = x\mathrm{e}^{g(\sqrt{x})}$. Find $f'(x)$.

$$f'(x) = \mathrm{e}^{g(\sqrt{x})} + x\mathrm{e}^{g(\sqrt{x})} \cdot g'(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2}$$

43. Find a formula for dy/dx: $x^2 + xy + y^3 = 0$.

$$2x + y + xy' + 3y^2y' = 0 \Rightarrow (x + 3y^2)y' = -(2x + y) \Rightarrow y' = \frac{-(2x + y)}{x + 3y^2}$$

44. Show that 5 is a critical number of $g(x) = 2 + (x - 5)^3$, but that $g$ does not have a local extremum there.

    $g'(x) = 3(x - 5)^2$, so $g'(5) = 0$.

    By looking at the sign of $g'(x)$ (First derivative test), we see that $g'(x)$ is always non-negative, so $g$ does not have a local min or max at $x = 5$.

45. Find the general antiderivative:

    (a) $f(x) = 4 - x^2 + 3\mathrm{e}^x$ $F(x) = 4x - \frac{1}{3}x^3 + 3\mathrm{e}^x + C$

    (b) $f(x) = \frac{3}{x^2} + \frac{2}{x} + 1$ $F(x) = -3x^{-1} + 2\ln|x| + x + C$

    (c) $f(x) = \frac{1+x}{\sqrt{x}}$

    First rewrite $f(x) = x^{-1/2} + x^{1/2}$, and

$$F(x) = 2x^{1/2} + \frac{2}{3}x^{3/2} + C$$

46. Find the slope of the tangent line to the following at the point (3,4): $x^2 + \sqrt{y}x + y^2 = 31$

$$2x + \frac{1}{2}y^{-1/2}y'x + \sqrt{y} + 2yy' = 0$$

At $x = 3, y = 4$:

$$6 + \frac{3}{4}y' + 2 + 8y' = 0 \quad \Rightarrow \quad y' = \frac{-32}{35}$$

$$y - 4 = \frac{-32}{35}(x - 3)$$

47. Find the critical values: $f(x) = |x^2 - x|$

    One way to approach this problem is to look at it piecewise. Use a table to find where $f(x) = x(x - 1)$ is positive or negative:

$$f(x) = \begin{cases} x^2 - x & \text{if } x \leq 0, \text{ or } x \geq 1 \\ -x^2 + x & \text{if } 0 < x < 1 \end{cases}$$

Now compute the derivative:

$$f'(x) = \begin{cases} 2x - 1 & \text{if } x < 0, \text{ or } x > 1 \\ -2x + 1 & \text{if } 0 < x < 1 \end{cases}$$

At $x = 0$, from the left, $f'(x) \to 1$ and from the right, $f'(x) \to -1$, so $f'(x)$ does not exist at $x = 0$.

At $x = 1$, from the left, $f'(x) \to -1$, and from the right, $f'(x) \to 1$, so $f'(x)$ does not exist at $x = 1$.

Finally, $f'(x) = 0$ if $2x - 1 = 0 \Rightarrow x = 1/2$, but $1/2$ is not in that domain. The other part is where $-2x + 1 = 0$, which again is $1/2$, and this time it is in $0 < x < 1$.

The critical points are: $x = 1/2, 0, 1$.

48. Does there exist a function $f$ so that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$ for all $x$?

$$f'(x) = \frac{f(2) - f(0)}{2 - 0} = \frac{5}{2}$$

Since $\frac{5}{2} > 2$, there can exist no function like that (that is continuous).

49. Linearize $f(x) = \sqrt{1 + x}$ at $x = 0$.

Point: $x = 0, y = 1$
Slope: $f'(0) = \frac{1}{2}$
Line: $y - 1 = \frac{1}{2}(x - 0)$, or $y = \frac{1}{2}x + 1$

50. Find $dy$ if $y = \sqrt{1 - x}$ and evaluate $dy$ if $x = 0$ and $dx = 0.02$. Compare your answer to $\Delta y$

$$dy = \frac{-1}{2\sqrt{1 - x}}\, dx, \quad \Rightarrow dy = \frac{1}{2\sqrt{1 - 0}} \cdot 0.02 = 0.01$$

$$\Delta y = \sqrt{1 - 0.02} - \sqrt{1} = 0.01005...$$

51. Fill in the question marks: If $f''$ is positive on an interval, then $f'$ is INCREASING and $f$ is CONCAVE UP.

52. If $f(x) = x - \cos(x)$, $x$ is in $[0, 2\pi]$, then find the value(s) of $x$ for which

(a) $f(x)$ is greatest and least.
Here we are looking for the maximum and minimum- use a table with endpoints and critical points. To find the critical points,

$$f'(x) = 1 + \sin(x) = 0 \Rightarrow \sin(x) = -1 \Rightarrow x = \frac{3\pi}{2}$$

is the only critical point in $[0, 2\pi]$.
Now the table:

| $x$ | $0$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|
| $f(x)$ | $-1$ | $\frac{3}{\pi}2 \approx 4.7$ | $2\pi - 1 \approx 5.2$ |

so $f$ is greatest at $x = 2\pi$, least at $0$.

(b) $f(x)$ is increasing most rapidly.

Another way to say this: Where's the maximum of $f'(x)$? We've computed $f'(x)$ to be: $1+\sin(x)$, so take its derivative: $\cos(x) = 0$. So there are two critical points at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. Checking these and the endpoints:

| $x$ | $0$ | $\frac{\pi}{2}$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|
| $f'(x)$ | 1 | 2 | 0 | 1 |

so $f$ is increasing most rapidly at $x = \frac{\pi}{2}$.

(c) The slopes of the lines tangent to the graph of $f$ are increasing most rapidly.

Another way to say this: Where is $f'(x)$ increasing most rapidly? At the maximum of $f''(x)$. The maximum of $\cos(x)$ in the interval $[0, 2\pi]$ occurs at $x = 0$ and $x = 2\pi$.

53. Show there is *exactly* one root to: $\ln(x) = 3 - x$

First, to use the Intermediate Value Theorem, we'll get a function that we can set to zero: Let $f(x) = \ln(x) - 3 + x$. Then a root to $\ln(x) = 3 - x$ is where $f(x) = 0$.

First, by plugging in numbers, we see that $f(2) < 0$ and $f(3) > 0$. There is at least one solution in the interval $[2, 3]$ by the Intermediate Value Theorem.

Now, is there more than one solution? $f'(x) = \frac{1}{x} + 1$ which is always positive for positive $x$. This means that, for $x > 0$, $f(x)$ is always increasing. Therefore, if it crosses the $x-$axis (and it does), then $f$ can never cross again.

54. Approximate the change in volume of a cone, if we assume the height to be constant and $r$ changes from 2 to 2.1. ($V = \frac{1}{3}\pi r^2 h$)

Approximation means to use $dV$, which is ($h$ is constant:)

$$dV = \frac{2}{3}\pi r h \, dr$$

Plug in the numbers, using $r = 2$ and $dr = 0.1$ to get

$$dV \approx 0.133\pi h$$

55. Sketch the graph of a function that satisfies all of the given conditions:

$$\begin{array}{ccc} f(1) = 5 & f(4) = 2 & f'(1) = f'(4) = 0 \\ \lim_{x \to 2^+} f(x) = \infty, & \lim_{x \to 2^-} f(x) = 3 & f(2) = 4 \end{array}$$

(We'll do this one in class)

56. If $s^2 t + t^3 = 1$, find $\frac{dt}{ds}$ and $\frac{ds}{dt}$.

First, treat $t$ as the function, $s$ as the variable:

$$2st + s^2 \frac{dt}{ds} + 3t^2 \frac{dt}{ds} = 0 \Rightarrow \frac{dt}{ds} = \frac{-2st}{s^2 + 3t^2}$$

For $s$ as the function, $t$ as the variable:

$$\frac{ds}{dt} = \frac{-(s^2 + 3t^2)}{2st}$$

which you can either state directly or show.

57. Antidifferentiate:

(a) $f'(x) = 3\sqrt{x} - \frac{1}{\sqrt{x}}$, $f(1) = 2$
First, rewrite:

$$f'(x) = 3x^{1/2} - x^{-1/2}$$

so that

$$f(x) = 3 \cdot \frac{2}{3} \cdot x^{3/2} + 2x^{1/2} + C$$

Now, $f(1) = 2$, so:

$$2 = 2 + 2 + c \Rightarrow c = -2$$

Finally,

$$f(x) = 2x^{3/2} + 2x^{1/2} - 2$$

(b) $f''(x) = x^2 + 3\cos(x), \quad f(0) = 2, f'(0) = 3$

$$f'(x) = \frac{1}{3}x^3 + 3\sin(x) + C$$

with $f'(0) = 3$, so $C = 3$. Now,

$$f'(x) = \frac{1}{3}x^3 + 3\sin(x) + 3$$

so that

$$f(x) = \frac{1}{3} \cdot \frac{1}{4} \cdot x^4 - 3\cos(x) + 3t + C = \frac{1}{12}x^4 - 3\cos(x) + 3t + C$$

and $f(0) = 2$, so

$$2 = -3 + C \Rightarrow C = 5$$

so that:

$$f(x) = \frac{1}{12}x^4 - 3\cos(x) + 3t + 5$$

(c) $f''(x) = 3e^x + 5\sin(x)$, $f(0) = 1$, $f'(0) = 2$
We'll do this one a bit differently, although we could do it like before:

$$f'(x) = 3e^x - 5\cos(x) + C_1$$

$$f(x) = 3e^x - 5\sin(x) + C_1 t + C_2$$

To find $C_1, C_2$, plug in the numbers: From $f'(0) = 2$,

$$2 = 3 - 5 + C_1 \Rightarrow C_1 = 4$$

To find $C_2$, we do something similar:

$$1 = 3 + 0 + 4 \cdot 0 + C_2 \Rightarrow C_2 = -2$$

so:

$$f(x) = 3e^x - 5\sin(x) + 4t - 2$$

(d) $f'(x) = \frac{4}{\sqrt{1-x^2}}, f(1/2) = 1$
We see that:

$$f'(x) = 4 \cdot \frac{1}{\sqrt{1-x^2}}$$

so that:

$$f(x) = 4\sin^{-1}(x) + C$$

Now, at $x = 1/2$,

$$1 = 4\sin^{-1}(1/2) + C \Rightarrow 1 = 4 \cdot \frac{\pi}{6} + C \Rightarrow C = 1 - \frac{2\pi}{3}$$

and $f(x) = 4\sin^{-1}(x) + \left(1 - \frac{2\pi}{3}\right)$