CALC I, EXAM I SAMPLE SOLUTIONS

1. True or False, and explain:

   (a) The derivative of a polynomial is a polynomial.
   True. A polynomial is a sum of terms like ax^n, and each derivative will be $a \cdot nx^{n-1}$. If $n \geq 1$ is a positive integer, so will $n-1$.

   (b) If f is continuous and $f(1) = 3$, $f(2) = 4$ then there is an r so that $f(r) = \pi$.
   True, by the Intermediate Value Theorem (IVT). In fact, r will be between 1 and 2.

   (c) If f is differentiable and positive, then

   $$\frac{d}{dx}\sqrt{f(x)} = \frac{d}{dx}\left(f(x)\right)^{1/2} = \frac{1}{2}(f(x))^{-1/2}f'(x)$$

   So the statement is True.

   (d) The equation of the tangent line to $y = x^3$ at $x = 1$ is $y - x = 3x^2(x-1)$.
   False. $3x^2$ is a formula for the slope, and not the slope itself. The tangent line is: $y - 1 = 3(x-1)$.

   (e) All functions are continuous on their domains.
   False. All of our basic functions (polynomials, trig functions, rational functions, algebraic functions) are continuous on their domain. The following function is not continuous on its domain:

   $$f(x) = \begin{cases} x^2 & \text{if } x \leq 1 \\ 3 - x & \text{if } x > 1 \end{cases}$$

   (f) If the limit is 5, then f is continuous at 5
   True. The statement is another way of saying that f is differentiable at $x = 5$. We know that if f is differentiable, then it is continuous.

   (g) If we get $\frac{0}{0}$...
   We cannot say if the limit exists or not. We must do additional analysis.

   (h) $f(x) = |x|$ is not continuous at $x = 0$.
   False. $|x|$ is continuous at $x = 0$ (but it is not differentiable at $x = 0$.

   (i) If $y = ax + b$ then $\frac{dy}{da} = x$.
   True. The notation $\frac{dy}{da}$ implies that y is a function of a, so that x and b are constants. Therefore, the derivative of $ax + b$ with respect to a is x.

   (j) If $f(x) = (3x^2 + 6x)^4$, then $f'(x) = 4(6x + 6)^3$.
   False. The chain rule is misapplied here. The derivative is:

   $$4(3x^2 + 6x)^3(6x + 6)$$

2. Finish the definitions:

   (a) A function f is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$

   (b) The derivative of f at $x = a$ is:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{or} \quad \lim_{v \to a} \frac{f(v) - f(a)}{v - a}$$

3. How would we use the product rule and chain rule to differentiate $h(x) = f(x)/g(x)$?

   First write as $h(x) = f(x) \cdot (g(x))^{-1}$. The derivative is then (by the product rule):

$$h'(x) = f'(x)(g(x))^{-1} + f(x) \cdot -1(g(x))^{-2} g'(x)$$

   Simplifying, we get the quotient rule:

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

4. The definition of continuity of f at $x = a$ implies that: (i) a is in the domain of f, (ii) The limit as $x \to a$ of f exists, (iii) the limit is the same as $f(a)$.

   For the graphs, see the website.

5. IVT: Let f be continuous on $[a, b]$ and let ν be any point between $f(a)$ and $f(b)$. Then there is a c in $[a, b]$ so that $f(c) = \nu$.

   For the graphs, see the website

6. The IVT is normally used to show the existence of solutions to equations, and in particular to show that a function has (at least) one zero. In this case, we would need to find a, b so that $f(a)$ and $f(b)$ have different signs.

   To use the IVT for $f(x) = g(x)$, first write as $f(x) - g(x) = 0$, then look for a, b so that $f(a) - g(a)$ and $f(b) - g(b)$ have different signs.

7. See the website for a sample picture.

8. This is a pattern recognition problem. Parts (a) and (c) are using the first definition of the derivative (see problem 2(b)), and part (b) is using the second definition.

   (a) $f(x) = \sqrt{x}$, $a = 1$.

   (b) $f(x) = \sqrt[9]{x}$, $a = 1$.

   (c) $f(x) = \frac{1}{x}$, $a = 3$.

9. An alternative way of stating this problem: Find the equation(s) of the tangent lines to $y = x^2$ that go through the additional point $(4, 15)$. If we find these values of x, since we're moving from left to right, we'd choose the smaller x.

   Let a be the $x-$coordinate we're looking for. Then the line goes through the points (a, a^2) and $(4, 15)$. This says that the slope of the line should be:

$$m = \frac{a^2 - 15}{a - 4}$$

   On the other hand, this is a tangent line to x^2, so the slope of the tangent line at $x = a$ should be the derivative, $2a$.

   Putting these together,

$$2a = \frac{a^2 - 15}{a - 4}$$

   Clear the fractions and simplify to get: $a^2 - 8a + 15 = 0$. The solutions are $a = 3, a = 5$. We choose the smaller value, $a = 3$, since we're moving from left to right.

10. To say that the tangent line is perpendicular to the given line means that the slopes are negative reciprocals of each other.

    In this case, the given slope is 2, so we want the derivative at some value a to be $\frac{-1}{2}$:

$$\frac{-1}{2} = 2(a - 2)$$

    so $a = \frac{7}{4}$.

11. This is a chain rule problem:

$$V = 3w^2 + 5w^{-1} \Rightarrow \frac{dV}{dw} = 6w - 5w^{-2} = 6w - \frac{5}{w^2}$$

$$w = (u + 1)^{1/2} \Rightarrow \frac{dw}{du} = \frac{1}{2}(u + 1)^{-1/2} = \frac{1}{2\sqrt{u + 1}}$$

$$u = t^2 + t \Rightarrow \frac{du}{dt} = 2t + 1$$

    Therefore,

$$\frac{dV}{dw} = 6w - \frac{5}{w^2}$$

$$\frac{dV}{du} = \frac{dV}{dw} \cdot \frac{dw}{du} = \left(6w - \frac{5}{w^2}\right) \frac{1}{2\sqrt{u + 1}} = \left(6\sqrt{u + 1} - \frac{5}{u + 1}\right) \frac{1}{2\sqrt{u + 1}}$$

$$= 3 - \frac{5}{(u + 1)^{3/2}}$$

$$\frac{dV}{dt} = \frac{dV}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dt} = \frac{dV}{du} \cdot \frac{du}{dt} = \left(3 - \frac{5}{(u + 1)^{3/2}}\right)(2t + 1) =$$

$$\frac{dV}{dt} = \left(3 - \frac{5}{(t^2+t+1)^{3/2}}\right)(2t+1)$$

12. Show that there is at least one solution to $x^5 = x^2 + 4$. Start by putting in a better form. We'll rewrite the equation to be:

$$x^5 - x^2 - 4 = 0$$

and now find x so that $f(x) = x^5 - x^2 - 4$ is positive, and where it is negative (this is by trial and error). For example, at $x = 1$, $f(1) = 1 - 1 - 4 < 0$, and $f(2) = 32 - 4 - 4 > 0$. Therefore there is at least one solution to $f(x) = 0$ between $x = 1$ and $x = 2$ (This is the Intermediate Value Theorem).

13. Find the equation of the tangent line to $f(x)$ at $x = a$. Recall that the equation of the tangent line can be written as:

$$y - a = f'(a)(x - a)$$

(a) Rewrite f so that we can easily differentiate:

$$f(x) = x + 2 + 5x^{-1} \Rightarrow f'(x) = 1 - \frac{5}{x^2}$$

so the slope at $x = 1$ is $f'(1) = 1 - 5 = -4$. The point is found by $(1, f(1)) = (1, 8)$. The tangent line is $y - 8 = -4(x - 1)$

(b) There are a couple of different approaches to differentiate. One way to do it is to rewrite:

$$f(u) = u^5 + 2u^4 + 3, \text{ and } u = x^2 + 3x$$

To differentiate,

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = (5u^4 + 8u^3)(2x + 3)$$

At $x = 0, u = 0$, so substituting these values in give a slope of zero. The equation of the tangent line is a horizontal line going through $(0, 3)$, therefore it is $y = 3$.

14. If $f(x) = x^2$, then $f(x + h) = (x + h)^2 = x^2 + 2xh + h^2$, and $f(x - 2h) = (x - 2h)^2 = x^2 - 4xh + 4h^2$. Putting these into the given expression,

$$\frac{x^2 + 2xh + h^2 - (x^2 - 4xh + 4h^2)}{3h} = \frac{6xh - 3h^2}{3h} = 2x - h$$

15. We could restate this problem as: Find the domain of f. In these cases, we want to use a sign chart to make sure that the expression under the square root sign is nonnegative.

(i) $x(4-x) \geq 0$:

x	$-$	$+$	$+$
$(x-4)$	$-$	$-$	$+$
	$x<0$	$0<x<4$	$x>4$

so altogether, $x(4-x) \geq 0$ if $x \leq 0$ or $x \geq 4$.

(ii) $\frac{x^2-4}{1-x^2} \geq 0$:

$x-2$	$-$	$-$	$-$	$-$	$+$
$x+2$	$-$	$+$	$+$	$+$	$+$
$1-x$	$+$	$+$	$+$	$-$	$-$
$1+x$	$-$	$-$	$+$	$+$	$+$
	$x<-2$	$-2<x<-1$	$-1<x<1$	$1<x<2$	$x>2$

So the domain is $-2 \leq x < -1$ and $1 < x \leq 2$

16. Rewrite as a piecewise defined function:

If $f(x) = \frac{|3x+2|}{3x+2}$, then

$$f(x) = \begin{cases} \frac{3x+2}{3x+2} & \text{if } 3x+2>0 \\ -\frac{3x+2}{3x+2} & \text{if } 3x+2<0 \end{cases} = \begin{cases} 1 & \text{if } x > -2/3 \\ -1 & \text{if } x < -2/3 \end{cases}$$

If $f(x) = \left|\frac{x-2}{(x+1)(x+2)}\right|$, then we need to compute where $\frac{x-2}{(x+1)(x+2)}$ is positive or negative. Build a sign chart:

$x-2$	$-$	$-$	$-$	$+$
$x+1$	$-$	$-$	$+$	$+$
$x+2$	$-$	$+$	$+$	$+$
	$x<-2$	$-2<x<-1$	$-1<x<2$	$x>2$

So

$$f(x) = \begin{cases} \frac{x-2}{(x+1)(x+2)} & \text{if } -2<x<-1 \text{ or } x \geq 2 \\ -\frac{x-2}{(x+1)(x+2)} & \text{if } x<-2 \text{ or } -1<x<2 \end{cases}$$

17. Compute the limit if it exists:

(a) $$\lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}} =$$

$$\lim_{h\to 0} \frac{1}{h} \cdot \frac{x-(x+h)}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})} = \frac{-1}{x \cdot 2\sqrt{x}} = \frac{-1}{2x^{3/2}}$$

(b) $$\lim_{x\to 2} \frac{x^2-4}{\frac{1}{x}-\frac{1}{2}} = \lim_{x\to 2} \frac{x^2-4}{\frac{2-x}{2x}} = \lim_{x\to 2} \frac{(x+2)(x-2)2x}{2-x} = \lim_{x\to 2} -(x+2)2x = -16$$

(c) $\lim_{x\to 2} \frac{x^2-2x-3}{x-3} = \frac{4-4-3}{2-3} = 3$

(d) $\lim_{x\to 3} \frac{1}{x-3}$ This limit does not exist. As $x \to 3$, the denominator is getting very small ($\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$, but since the numerator is constant, the fraction overall is going to $\pm\infty$ $(10, 100, 1000, \ldots)$. Recall that in our extra practice sheet we considered what happens for $k/0$ versus $0/0$ versus $0/k$.

(e) $\lim_{x\to 0} x^2 \sin(A)$ I don't care what A is, as long as it is going to infinity and not a number. If A were going to a number as $x \to 0$, then the limit could be found by substitution of $x = 0$. Since A is going to infinity, we can use the Squeeze Theorem:

$$-x^2 \le x^2 \sin(A) \le x^2$$

Because the left and right sides of the inequality go to zero, so must the middle function. (The answer is zero, by the squeeze theorem).

(f) $\lim_{x\to 2} \frac{x^2-4}{2x^2+3x-14} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(2x+7)(x-2)} = \frac{4}{11}$
If you had trouble factoring the denominator, you can also use long division since we know that $(x-2)$ factors out of it. Alternatively, one could also use the quadratic formula to get the factors.

(g) $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|}\right)$
Before continuing, rewrite the function to get rid of the absolute values:

$$\frac{1}{x} - \frac{1}{|x|} = \begin{cases} 0 & \text{if } x > 0 \\ \frac{2}{x} & \text{if } x < 0 \end{cases}$$

so the limit as $x \to 0^+$ is 0 (we're only considering the top part of the function).

18. Find all values of a for which f is continuous: In these problems, explicitly check the three things for continuity.

(a) $f(x) = \begin{cases} 4-x^2 & \text{if } x \le 3 \\ 3-ax & \text{if } x > 3 \end{cases}$ (i) $f(3) = -5$, (ii) For the limit, we have to take each side, we which do in a moment. But notice that, since $f(3)$ is part of the upper function, if the limit exists, the function value will automatically be at the limit, so we won't have to check (iii) specifically.

$$\lim_{x\to 3^-} 4-x^2 = -5, \qquad \lim_{x\to 3^+} 3-ax = 3-3a$$

so $3-3a = -5$, or $a = 8/3$. For this value of a, the limit and the functional value at 3 is -5.

(b) $f(x) = \begin{cases} x^2 - 2 & \text{if } x \le a \\ 2x - 1 & \text{if } x > a \end{cases}$ Again, go through our three parts: (i) $f(a) = a^2 - 2$, which exists for all a. (ii) For the limit, take each one separately:

$$\lim_{x \to a^-} x^2 - 2 = a^2 - 2, \qquad \lim_{x \to a^+} 2x - 1 = 2a - 1$$

For the limit to exist, $a^2 - 2 = 2a - 1 \Rightarrow a^2 - 2a - 1 = 0$. Use the quadratic formula to get that $a = 1 - \sqrt{2}$ or $a = 1 + \sqrt{2}$. At these values of a, the limit exists. Furthermore, the limit is equal to $a^2 - 2$ which is the functional value at a.

(c) $f(x) = \begin{cases} \frac{x^2-16}{x^2-4} & \text{if } x \neq \pm 2 \\ a & \text{if } x = \pm 2 \end{cases}$ Since $x - 2$ and $x + 2$ do not factor out of the denominator, there is no value of a that will make this function continuous.

19. Find all values of a so that the limit exists, then compute the limit: Use long division to get that

$$\frac{x^3 - 2x^2 + 2x + a}{x - 3} = x^2 + x + 5 + \frac{a + 15}{x - 3}$$

For the limit to exist, $x - 3$ must factor out of the numerator, so $a = -15$. In that case, the limit is $\frac{3^2+3+5}{3} = \frac{17}{3}$ (there was still an x left over in the denominator).

20. Compute $f'(x)$ using the definition:

$$\lim_{h \to 0} \frac{\frac{x+h}{x+h+1} - \frac{x}{x+1}}{h} = \lim_{h \to 0} \frac{1}{h} \cdot \frac{(x+h)(x+1) - x(x+h+1)}{(x+h+1)(x+1)} =$$

$$\lim_{h \to 0} \frac{1}{(x+h+1)(x+1)} = \frac{1}{(x+1)^2}$$

21. Compute $h'(w)$ using the definition:

$$\lim_{h \to 0} \frac{\frac{3}{w+h} - \frac{3}{w}}{h} = 3 \lim_{h \to 0} \frac{1}{h} \cdot \frac{w - (w+h)}{w(w+h)} = 3 \lim_{h \to 0} \frac{-1}{w(w+h)} = \frac{-3}{w^2}$$

22. Compute the velocity if position is $\sqrt{t}$ by using the definition of the derivative:

$$\lim_{h \to 0} \frac{\sqrt{t+h} - \sqrt{t}}{h} \cdot \frac{\sqrt{t+h} + \sqrt{t}}{\sqrt{t+h} + \sqrt{t}} = \lim_{h \to 0} \frac{1}{\sqrt{t+h} + \sqrt{t}} = \frac{1}{2\sqrt{t}}$$

23. Compute the derivative using a shortcut rule:

(a) $y(x) = 5x^{6/5}$, $y'(x) = 5 \cdot \frac{6}{5} x^{1/5} = 6x^{1/5}$

(b) $f(x) = 3x^2 - 5x^{1/3} - \frac{1}{3}x^{-1}$, $f'(x) = 6x - \frac{5}{3}x^{-2/3} + \frac{1}{3}x^{-2}$

(c) $h(x) = (3x^2 + 15x - 9)^{4/3}$ Use the Chain Rule:

$$h'(x) = \frac{4}{3}(3x^2 + 15x - 9)^{1/3}(6x + 15)$$

(d) $g(s) = 2(2 - 3s)^{-1/2}$, $g'(s) = 2 \cdot -\frac{1}{2}(2 - 3s)^{-3/2}(-3)$,

$$g'(s) = 3(3 - 2s)^{-3/2}$$

(e) $h(t) = \frac{3t-5}{t^2-6t}$ Use the Quotient Rule:

$$h'(t) = \frac{3(t^2 - 6t) - (3t - 5)(2t - 6)}{(t^2 - 6t)^2} = -\frac{3t^2 - 10t + 30}{(t^2 - 6t)^2}$$

(f) $f(u) = (3x + 5)\sqrt{2x + 1}$. This should read $f(x)$ rather than $f(u)$, otherwise $f'(u) = 0$. Given that typo, the derivative is (use the Product rule and Chain Rule):

$$f'(x) = 3\sqrt{2x + 1} + (3x + 5) \cdot \frac{1}{2}(2x + 1)^{-1/2} \cdot 2 = 3\sqrt{2x + 1} + \frac{3x + 5}{\sqrt{2x + 1}}$$

(g) $p(t) = x^2(2x - 7)^3$. This should read $p(x)$, otherwise $p'(t) = 0$. Given that, use the product and chain rules:

$$p'(x) = 2x(2x - 7)^3 + x^2 \cdot 3(2x - 7)^2 \cdot 2 = 2x(2x - 7)^3 + 6x^2(2x - 7)^2 =$$

$$(2x - 7)^2(2x(2x - 7) + 6x^2) = (2x - 7)^2 2x(5x - 7)$$

Final note on differentiation: Usually try to clean up the derivative somewhat- sometimes its hard to know what "simplify" means- a factored form is always best if you can get it. Don't spend too much time on the exam doing simplification- the bulk of the points will be in computing the derivative itself.