Summary Page: Exam III
Calculus I
Fall 2003

TOPICS:

- Functions and Algebraic Rules: Exponentials and Logarithms.
- Differentiation using Exponentials and Logarithms. (Be sure to memorize our full chart). Also: Differentiation of $(f(x))^{g(x)}$ using the exponential function. Be sure to distinguish between a^x, x^a and more generally $a^{f(x)}$ and $(f(x))^a$ and $(f(x))^{g(x)}$
- Differential Equations and Applications.
  1. If $y' = ky$, then $y(t) = A\mathrm{e}^{kt}$
  2. This model is used for population and radioactive decay.
  3. If $(f(t) - c)' = k(f(t) - c)$, then $f(t) = A\mathrm{e}^{kt} + c$
  4. Solve for A by using extra information like $f(a) = b$.
  5. Set up for Mixing Problems: If $A(t)$ is the amount at time t, then

     $$\frac{dA}{dt} = \left[\begin{array}{c}\text{Rate going}\\ \text{in}\end{array}\right] - \left[\begin{array}{c}\text{Rate going}\\ \text{out}\end{array}\right]$$

     Be sure to pay attention to the units used- it can help you figure out what you need. For example, the units of $\frac{dA}{dt}$ might be in $\frac{\text{pounds}}{\text{minute}}$.
- Limits at Infinity: Horizontal and Vertical Asymptotes.

  Definitions:

  - The function $f(x)$ has a horizontal asymptote at $y = L$ iff $\lim_{x\to\pm\infty} f(x) = L$
  - The function $f(x)$ has a vertical asymptote at $x = a$ iff $\lim_{x\to a^{+,-}} f(x) = \pm\infty$

  Techniques we learned: Divide by x^n, "Rationalize", Exponentials/Logs
- L'Hospital's Rule for taking limits: If the limit is of the form $\frac{0}{0}$ or $\pm\frac{\infty}{\infty}$, then:

  $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

  - This is NOT the quotient rule.
  - Algebra may be involved to get your expression into the right form: Divide or exponentiate.

Review Questions, Exam 3

1. True or False, and give a short reason:
   (a) $\ln(3+5) = \ln(3) + \ln(5)$
   (b) $\log_2(3) = \frac{\ln(3)}{\ln(2)}$
   (c) $e^{x-2} = e^x - e^2$
   (d) The equation of the tangent line to $y = e^x$ at $(1, e)$ is $y - e = e^x(x-1)$
   (e) $\lim_{x\to\infty} \sqrt{x^2+2x+4} - x = 0$
   (f) $\frac{d}{dx}10^x = x10^{x-1}$
   (g) There is a vertical asymptote at $x = 2$ for $\frac{\sqrt{x^2+5}-3}{x^2-2x}$

2. Give a general formula for the derivative of each. Assume that $a > 0$ is a constant.
   (a) $y = a^{f(x)}$
   (b) $y = \log_a(f(x))$
   (c) $y = e^{f(x)}$
   (d) $y = e^{f(x)g(x)}$
   (e) $y = f(x)^{g(x)}$
   (f) $y = \tan^{-1}(f(x))$

3. A tank initially contains 400 gallons of brine in which 100 gallons of salt are dissolved. Pure water is running into the tank at a rate of 20 gallons per minute, and the well stirred mixture is being drained off at the same rate. How many pounds of salt remain in the tank after 30 minutes?

4. Same as the previous problem, except that instead of pure water, brine containing $\frac{1}{10}$ pounds per gallon is run into the tank. How many pounds remain after 30 minutes?

5. A bacterial culture, growing exponentially, increases from 100 to 400 grams in 10 hours. How much was present after 3 hours?

6. Find the limit, if it exists.
   (a) $\lim_{x\to 0} \frac{3^x - 2^x}{x}$
   (b) $\lim_{x\to\infty} \frac{3x^2-5x+4}{\sqrt{x^4+5}}$
   (c) $\lim_{x\to\infty} \sqrt{x^2-3x+5} - x$
   (d) $\lim_{x\to\infty} \frac{\ln(1+e^x)}{1+x}$
   (e) $\lim_{x\to\infty} \left(1+\frac{4}{x}\right)^{2x}$
   (f) $\lim_{x\to 2^+} \frac{x+1}{x^2+x-6}$

7. Find all vertical and horizontal asymptotes for each function:
   (a) $f(x) = \frac{2x+3}{\sqrt{x^2-2x-3}}$
   (b) $f(x) = \sqrt{x+1} - x$
   (c) $f(x) = \frac{x^2-5x+6}{x-3}$
   (d) $f(x) = \frac{4x-5}{3x+2}$

8. A 300 gallon tank contains 100 gallons of brine with a concentration of one pound of salt per gallon or water. A brine containing $\frac{1}{2}$ pounds of salt per gallon runs into the tank at the rate of four gallons per minute, and the well stirred mixture runs out of the tank at the same rate. When in the concentration in the tank 0.7 pounds per gallon?

9. Find the minimum value of $y = e^x - e^{-2x}$.

10. Find all intervals on which f is increasing or decreasing: $f(x) = x^2 e^{-\frac{x}{4}}$

11. Rewrite using the rules of logarithms and exponents:

    (a) $2^{\log_2(x-5)-2}$

    (b) $2\ln(x) - 3\ln(x-2) + \frac{1}{2}\ln(5)$

    (c) $\ln(3e^{-5x+2})$

    (d) $\ln(3 + e^{-5x+2})$

    (e) $-\frac{2}{3}\log_5(5m^2) + \frac{1}{2}\log_5(25m^2)$

12. Write the exponential expression as an equivalent logarithm, or write the logarithm as an equivalent exponential.

    (a) $\log_{\sqrt{3}}(81) = 8$

    (b) $(2/3)^{-3} = 27/8$

    (c) $\log_2(r) = y$

13. Compute each of the following quantities on your calculator (4 decimal places is usually good enough):

    (a) $\log_{\sqrt{3}}(5)$

    (b) $5^{\log_2(3)}$

    (c) $\log_2(56)$

14. Solve for x:

    (a) $\ln(3x+2) + \ln(x-1) = 1$

    (b) $\ln(x^2) = (\ln(x))^2$

    (c) $3^{2x-1} = 4^{x+2}$

    (d) $\dfrac{4e^{2x}}{2 + e^{2x}} = 1$

    (e) $\ln(8x-7) - \ln(3+4x) = \ln(9/11)$

15. Differentiate:

    (a) $y = x^2 + 4 - \ln(3x)$

    (b) $y = 5^x - \log_3(x)$

    (c) $y = 2^{3^x}$

    (d) $y = 2^{\log_4(x)}$

    (e) $y = 7^{\tan(x)} - \log_2(x^2 - 3x + 5)$

    (f) $y = 3^{\arcsin(x)}$

    (g) $y = \log_7(x^2 - 3) + \frac{\log_4(x)}{1+x^2}$

    (h) $y = x^{3x}$

    (i) $y = \pi^{e^{2^x}}$

    (j) $y = (1 + x^2)^x$