Review Solutions, Exam 3

1. True or False, and give a short reason:

   (a) $\ln(3+5) = \ln(3) + \ln(5)$
   False. We do not have a rule for $\ln(a+b)$, but we could write $\ln(a) + \ln(b)$ as $\ln(ab)$.

   (b) $\log_2(3) = \frac{\ln(3)}{\ln(2)}$
   True. We could show this by:

   $$\log_2(3) = c \Leftrightarrow 2^c = 3 \Leftrightarrow \log(2^c) = \log(3) \Leftrightarrow c = \frac{\log(3)}{\log(2)}$$

   where log is for any base.

   (c) $e^{x-2} = e^x - e^2$
   False. We could write e^{x-2} as $e^x e^{-2}$

   (d) The equation of the tangent line to $y = e^x$ at $(1, e)$ is $y - e = e^x(x-1)$
   False. The derivative of e^x is e^x, but the slope of the tangent line at $x = 1$ is e^1, or just e.

   (e) $\lim_{x\to\infty} \sqrt{x^2+2x+4} - x = 0$ To obtain this limit, we "rationalize":

   $$\lim_{x\to\infty} \sqrt{x^2+2x+4} - x \cdot \frac{\sqrt{x^2+2x+4}+x}{\sqrt{x^2+2x+4}+x} = \lim_{x\to\infty} \frac{x^2+2x-4-x^2}{\sqrt{x^2+2x+4}+x} =$$

   $$\lim_{x\to\infty} \frac{2-\frac{4}{x}}{\sqrt{1+\frac{2}{x}+\frac{4}{x^2}}+1} = 1$$

   (f) $\frac{d}{dx} 10^x = x10^{x-1}$ False. We need to use the formula for exponential functions, which gives

   $$10^x \ln(10)$$

   (g) There is a vertical asymptote at $x = 2$ for $\frac{\sqrt{x^2+5}-3}{x^2-2x}$
   False. The limit is:

   $$\lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x^2-2x} \cdot \frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3} = \lim_{x\to 2} \frac{x^2+5-9}{x(x-2)(\sqrt{x^2+5}+3)} =$$

   $$\lim_{x\to 2} \frac{(x+2)(x-2)}{x(x-2)(\sqrt{x^2+5}+3)} = \lim_{x\to 2} \frac{(x+2)}{x(\sqrt{x^2+5}+3)} = \frac{4}{2(3+3)} = \frac{1}{3}$$

2. Give a general formula for the derivative of each. Assume that $a > 0$ is a constant.

(a) $y = a^{f(x)}$ $\qquad y' = a^{f(x)}\ln(a)\cdot f'(x)$

(b) $y = \log_a(f(x))$ $\qquad y' = \dfrac{f'(x)}{f(x)\ln(a)}$

(c) $y = \mathrm{e}^{f(x)}$ $\qquad y' = f'(x)\mathrm{e}^{f(x)}$

(d) $y = \mathrm{e}^{f(x)g(x)}$ $\qquad y' = \mathrm{e}^{f(x)g(x)}\left(f'(x)g(x) + f(x)g'(x)\right)$

(e) $y = f(x)^{g(x)}$ First, we write $(f(x))^{g(x)}$ as $\mathrm{e}^{g(x)\ln(f(x))}$. Now differentiate using the chain rule:

$$\frac{d}{dx}(f(x))^{g(x)} = \mathrm{e}^{g(x)\ln(f(x))}\left(g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}\right)$$

which we can rewrite as:

$$\frac{d}{dx}(f(x))^{g(x)} = (f(x))^{g(x)}\left(g'(x)\ln(f(x)) + \frac{f'(x)g(x)}{f(x)}\right)$$

NOTE: Do not memorize this formula- rather, remember how we get it.

(f) $y = \tan^{-1}(f(x))$

$$y' = \frac{f'(x)}{1 + (f(x))^2}$$

3. A tank initially contains 400 gallons of brine in which 100 gallons of salt are dissolved. Pure water is running into the tank at a rate of 20 gallons per minute, and the well stirred mixture is being drained off at the same rate. How many pounds of salt remain in the tank after 30 minutes?

   The general idea is to set this up as:

$$\frac{dA}{dt} = \begin{bmatrix}\text{Rate going}\\ \text{in}\end{bmatrix} - \begin{bmatrix}\text{Rate going}\\ \text{out}\end{bmatrix}$$

   Which in this case gives: $A(0) = 100$, and

$$\frac{dA}{dt} = 0 - \frac{A(t)\text{ pounds}}{400\text{ gallons}}\cdot\frac{20\text{ gallons}}{1\text{ minute}} = -\frac{1}{20}A(t)$$

   This gives $A(t) = A_0\mathrm{e}^{-\frac{1}{20}t}$. Using $A(0) = 100$, we get a final form, $A(t) = 100\mathrm{e}^{-\frac{1}{20}t}$. In 30 minutes, we'll have:

$$A(30) = 100\mathrm{e}^{-\frac{3}{2}} \approx 22.31\text{ pounds}$$

4. Same as the previous problem, except that instead of pure water, brine containing $\frac{1}{10}$ pounds per gallon is run into the tank. How many pounds remain after 30 minutes?

The setup is similar. Let $A(t)$ be the amount (in pounds) after t minutes. Then:

$$\frac{dA}{dt} = \frac{\frac{1}{10}\text{ pound}}{1\text{ gallon}} \cdot \frac{4\text{ gallons}}{1\text{ minute}} - \frac{A(t)\text{ pounds}}{400\text{ gallons}} \cdot \frac{20\text{ gallons}}{1\text{ minute}} = \frac{2}{5} - \frac{1}{20}A(t)$$

Now we rewrite to get into a standard form:

$$A'(t) = -\frac{1}{20}\left(A(t) - 8\right)$$

which can be rewritten as:

$$\left(A(t) - 8\right)' = -\frac{1}{20}\left(A(t) - 8\right)$$

Therefore, $A(t) = A_1 e^{-\frac{1}{20}t} + 8$. With $A(0) = 100$, we can find A_1:

$$100 = A_1 + 8 \Rightarrow A_1 = 92$$

so the final form is $A(t) = 92e^{-\frac{1}{20}t} + 8$. The quantity after 30 minutes will be:

$$A(30) = 92e^{-\frac{3}{2}} + 8 \approx 28.53\text{ pounds}$$

5. A bacterial culture, growing exponentially, increases from 100 to 400 grams in 10 hours. How much was present after 3 hours?

   We can assume that the population at time zero is 100. This will give the the population as:

$$P(t) = 100e^{kt}$$

   where $P(t)$ is the grams of bacteria after t hours. If at 10 hours, we have 400 grams, then we can solve for k:

$$400 = 100e^{10k} \Rightarrow 4 = e^{10k} \Rightarrow k = \frac{\ln(4)}{10} \approx 0.1386$$

   After 3 hours, we have:

$$P(3) = 100e^{\frac{3\ln(4)}{10}} \approx 151.57\text{ grams}$$

6. Find the limit, if it exists.

   (a) $\displaystyle\lim_{x\to 0} \frac{3^x - 2^x}{x} = \lim_{x\to 0} \frac{3^x \ln(3) - 2^x \ln(2)}{1} = \ln(3) - \ln(2)$ (by l'Hospital's rule)

   (b) $\displaystyle\lim_{x\to\infty} \frac{3x^2 - 5x + 4}{\sqrt{x^4 + 5}} = \lim_{x\to\infty} \frac{\frac{3x^2-5x+4}{x^2}}{\sqrt{\frac{x^4+5}{x^4}}} = \lim_{x\to\infty} \frac{3 - \frac{5}{x} + \frac{4}{x^2}}{\sqrt{1 + \frac{5}{x^4}}} = 3$ (Use $x^2 = \sqrt{x^4}$)

(c) $\lim_{x\to\infty} \sqrt{x^2-3x+5} - x \cdot \frac{\sqrt{x^2-3x+5}+x}{\sqrt{x^2-3x+5}+x} = \lim_{x\to\infty} \frac{x^2-3x+5-x^2}{\sqrt{x^2-3x+5}+x} =$
$\lim_{x\to\infty} \frac{-3x+5}{\sqrt{x^2-3x+5}+x} = \lim_{x\to\infty} \frac{-3+\frac{5}{x}}{\sqrt{1-\frac{3}{x}+\frac{5}{x^2}}+1} = -\frac{3}{2}$

(d) $\lim_{x\to\infty} \frac{\ln(1+e^x)}{1+x} = \lim_{x\to\infty} \frac{\frac{e^x}{1+e^x}}{1} = \lim_{x\to\infty} \frac{\frac{e^x}{e^x}}{\frac{1}{e^x}+\frac{e^x}{e^x}} = \frac{1}{0+1} = 1$

(e) $\lim_{x\to\infty} \left(1+\frac{4}{x}\right)^{2x}$ First, rewrite (like we did for derivatives of functions of this type):

$$\lim_{x\to\infty} \left(1+\frac{4}{x}\right)^{2x} = e^{\lim_{x\to\infty} 2x \ln\left(1+\frac{4}{x}\right)}$$

Get this in a form for l'Hospital:

$$\lim_{x\to\infty} 2x \ln\left(1+\frac{4}{x}\right) = 2 \lim_{x\to\infty} \frac{\ln\left(1+\frac{4}{x}\right)}{\frac{1}{x}}$$

Note that:

$$\frac{d}{dx} \ln\left(1+\frac{4}{x}\right) = \frac{1}{1+\frac{4}{x}} \cdot -\frac{4}{x^2} = \frac{1}{\frac{x+4}{x}} \cdot -\frac{4}{x^2} = \frac{-4}{x(x+4)}$$

so that l'Hospital's rule has the form:

$$2 \lim_{x\to\infty} \frac{\frac{-4}{x(x+4)}}{-\frac{1}{x^2}} = 2 \lim_{x\to\infty} \frac{4x^2}{x(x+4)} = 2 \cdot 4 = 8$$

Overall, the limit is:

$$\lim_{x\to\infty} \left(1+\frac{4}{x}\right)^{2x} = e^{\lim_{x\to\infty} 2x \ln\left(1+\frac{4}{x}\right)} = e^8$$

(f) $\lim_{x\to 2^+} \frac{x+1}{x^2+x-6}$ Notice that the numerator goes to 3 while the denominator goes to zero. Therefore, the overall limit goes to positive infinity.

7. Find all vertical and horizontal asymptotes for each function:

   To find vertical asymptotes: If the function is a fraction, then look for where the denominator is zero *and* be sure that the numerator is *not* zero there as well. Remember our general definition: $x = a$ is a vertical asymptote iff $\lim_{x\to a^\pm} f(x) = \pm\infty$

(a) $f(x) = \dfrac{2x+3}{\sqrt{x^2-2x-3}}$ In this case, the denominator factors $(x-3)(x+1)$. Note that the numerator is NOT zero when $x = 3$ or $x = -1$, therefore there are two vertical asymptotes, at $x = 3$ and $x = -1$.

Additional note: If this problem had been

$$f(x) = \frac{x-3}{\sqrt{x^2-2x-3}} = \frac{(x-3)}{(x-3)^{1/2}(x+1)^{1/2}} = \frac{\sqrt{x-3}}{\sqrt{x+1}}$$

there would not have been a vertical asymptote at $x = 3$ (we would have been able to factor it out).

To compute horizontal asymptotes, take the limit as $x \to \pm\infty$

$$\lim_{x\to\infty} \frac{2x+3}{\sqrt{x^2-2x-3}} = \lim_{x\to\infty} \frac{2+\frac{3}{x}}{\frac{\sqrt{x^2-2x-3}}{x}} = \lim_{x\to\infty} \frac{2+\frac{3}{x}}{\sqrt{\frac{x^2-2x-3}{x^2}}} = \lim_{x\to\infty} \frac{2+\frac{3}{x}}{\sqrt{1-\frac{2}{x}-\frac{3}{x^2}}} = 2$$

Note that we used the fact that, if $x > 0$, then $x = \sqrt{x^2}$. When we take the limit as x approaches $-\infty$, this quantity changes to: If $x < 0$, then $x = -\sqrt{x^2}$

$$\lim_{x\to-\infty} \frac{2x+3}{\sqrt{x^2-2x-3}} = \lim_{x\to-\infty} \frac{2+\frac{3}{x}}{\frac{\sqrt{x^2-2x-3}}{x}} = \lim_{x\to-\infty} \frac{2+\frac{3}{x}}{-\sqrt{\frac{x^2-2x-3}{x^2}}} = \lim_{x\to-\infty} \frac{2+\frac{3}{x}}{-\sqrt{1-\frac{2}{x}-\frac{3}{x^2}}} = -2$$

Now we have two horizontal asymptotes, $y = \pm 2$.

(b) $f(x) = \sqrt{x+1} - x$

There are no vertical asymptotes. For horizontal asymptotes, take the limit as $x \to \infty$. We cannot take the limit as $x \to -\infty$, since this is outside the domain.

$$\lim_{x\to\infty} \sqrt{x+1} - x \cdot \frac{\sqrt{x+1}+x}{\sqrt{x+1}+x} = \lim_{x\to\infty} \frac{x+1-x^2}{\sqrt{x+1}+x} = \lim_{x\to\infty} \frac{1+\frac{1}{x}-x}{\sqrt{\frac{1}{x}+\frac{1}{x^2}}+1} = -\infty$$

There are no horizontal asymptotes.

(c) $f(x) = \dfrac{x^2-5x+6}{x-3}$

The numerator factors to: $(x-2)(x-3)$, so

$$f(x) = \begin{cases} x-2, & \text{if } x \neq 3 \\ \text{not defined} & \text{if } x = 3 \end{cases}$$

So, again there are no vertical or horizontal asymptotes.

(d) $f(x) = \dfrac{4x-5}{3x+2}$ This is a "generic" case. There is a vertical asymptote at $x = -\frac{2}{3}$, and a horizontal asymptote (for both $\pm\infty$) at $y = \frac{4}{3}$

8. A 300 gallon tank contains 100 gallons of brine with a concentration of one pound of salt per gallon or water. A brine containing $\frac{1}{2}$ pounds of salt per gallon runs into the tank at the rate of four gallons per minute, and the well stirred mixture runs out of the tank at the same rate. When in the concentration in the tank 0.7 pounds per gallon?

   SOLUTION: Let $Q(t)$ be the pounds of salt in the tank at time t. Given the first sentence, there is initially 100 pounds of salt, so we know that $Q(0) = 100$.

   Now, the rate of change is given by:

$$Q' = \frac{4 \text{ gal}}{1 \text{ min}} \cdot \frac{\frac{1}{2} \text{ pound}}{1 \text{ gal}} - \frac{4 \text{ gal}}{1 \text{ min}} \cdot \frac{Q \text{ pounds}}{100\text{gal}} = 2 - \frac{1}{25}Q$$

   Therefore,

$$(Q - 50)' = -\frac{1}{25}(Q - 50)$$

   so that

$$Q(t) = A\mathrm{e}^{-\frac{1}{25}t} + 50$$

   Use the initial condition to solve for A,

$$100 = A + 50$$

   so that $A = 50$. Now, we have the complete solution to the differential equation,

$$Q(t) = 50\mathrm{e}^{-\frac{1}{25}t} + 50$$

   When is the concentration 0.7? This would mean that there is: $0.7 \cdot 100 = 70$ pounds of salt:

$$70 = 50\mathrm{e}^{-\frac{1}{25}t} + 50 \quad \Rightarrow \quad \frac{2}{5} = \mathrm{e}^{-\frac{1}{25}t}$$

   so that:

$$\ln(2/5) = -\frac{1}{25}t \quad \Rightarrow \quad t = -25\ln(2) \approx 22.91$$

   It takes approximate 22.91 minutes for the tank to reach the desired concentration.

9. Find the minimum value of $y = \mathrm{e}^x + \mathrm{e}^{-2x}$.

   NOTE: TYPO in the original question- The original question was subtracting the functions rather than adding them**

   We find the critical points by setting the derivative equal to zero:

$$y = \mathrm{e}^x + 2\mathrm{e}^{-2x} \quad \Rightarrow \quad y' = \mathrm{e}^x - 2\mathrm{e}^{-2x} = 0 \quad \Rightarrow \quad \mathrm{e}^x = 2\mathrm{e}^{-2x} \quad \Rightarrow \quad \mathrm{e}^{3x} = 2$$

   so that $x = \frac{\ln(2)}{3} \approx 0.231$. By using test points ($x = 0$ and $x = 1$, for example), we see that f is always decreasing before $x = \frac{\ln(2)}{3}$, and always increasing afterwards. Therefore, there is a global minimum at this value of x.

10. Find all intervals on which f is increasing or decreasing: $f(x) = x^2 e^{-\frac{x}{4}}$

We need to determine where f' is positive/negative. Take the derivative and use a sign chart:

$$f'(x) = 2x e^{-\frac{x}{4}} - x^2 \cdot \frac{-1}{4} e^{\frac{-x}{4}} = e^{-x/4} \cdot x \left(2 - \frac{1}{4}x\right)$$

Note that $e^{-x/4}$ is always positive. Therefore, we only need to check the other terms.

x	$-$	$+$	$+$
$(2 - x/4)$	$+$	$+$	$-$
	$x < 0$	$0 < x < 8$	$x > 8$

Overall, the function is increasing if $0 < x < 8$, and decreasing otherwise.

11. Rewrite using the rules of logarithms and exponents:

(a) $2^{\log_2(x-5)-2} = 2^{\log_2(x-5)} 2^{-2} = \frac{1}{4}(x-5)$

(b) $2\ln(x) - 3\ln(x-2) + \frac{1}{2}\ln(5) = \ln\left(\frac{x^2\sqrt{5}}{(x-2)^3}\right)$

(c) $\ln(3e^{-5x+2}) = \ln(3) + \ln(e^{-5x+2}) = \ln(3) - 5x + 2$

(d) $\ln(3 + e^{-5x+2})$. No simplification possible

(e) $-\dfrac{2}{3}\log_5(5m^2) + \dfrac{1}{2}\log_5(25m^2)$

$$\log_5((5m^2)^{-2/3}) + \log_5(5m) = \log_5(5^{-2/3}5^1 m^{-4/3} m^1) = \log_5(5^{1/3} m^{-1/3})$$

$$= \frac{1}{3} - \frac{1}{3}\log_5(m)$$

12. Write the exponential expression as an equivalent logarithm, or write the logarithm as an equivalent exponential.

(a) $\log_{\sqrt{3}}(81) = 8 \quad \Rightarrow \quad (\sqrt{3})^8 = 81 \quad \Rightarrow \quad 3^4 = 81$

(b) $(2/3)^{-3} = 27/8 \quad \Rightarrow \quad \log_{2/3}(27/8) = -3$ or $-3 = \dfrac{\ln(27/8)}{\ln(2/3)}$

(c) $\log_2(r) = y \quad \Rightarrow \quad 2^y = r$

13. Compute each of the following quantities on your calculator (4 decimal places is usually good enough):

(a) $\log_{\sqrt{3}}(5) \quad \Rightarrow \quad \dfrac{\ln(5)}{\frac{1}{2}\ln(3)} \approx 2.9299$

(b) $5^{\log_2(3)}$ Compute the exponent first: $\log_2(3) = \frac{\ln(3)}{\ln(2)} \approx 1.58496$, so the final approximation is 12.8186

(c) $\log_2(56) = \frac{\ln(56)}{\ln(2)} \approx 5.8074$

14. Solve for x:

 (a) $\ln(3x+2) + \ln(x-1) = 1$

 $$\ln((3x+2)(x-1)) = 1 \quad \Rightarrow \quad (3x+2)(x-1) = \mathrm{e} \quad \Rightarrow \quad 3x^2 - x - (2+\mathrm{e}) = 0$$

 $$x = \frac{1 \pm \sqrt{1 + 12(2+e)}}{6} \approx 1.432, -1.0984$$

 We disregard the second solution, which would make $x - 1 < 0$

 (b) $\ln(x^2) = (\ln(x))^2$

 $$(\ln(x))^2 - 2\ln(x) = 0 \quad \Rightarrow \quad \ln(x)\,(\ln(x) - 2) = 0$$

 So either $\ln(x) = 0$ (in which case $x = \mathrm{e}^0 = 1$) or $\ln(x) = 2$ (in which case $x = \mathrm{e}^2$).

 (c) $3^{2x-1} = 4^{x+2}$. Take the log of both sides (any base will do, but we normally would choose ln):

 $$\ln\left(3^{2x-1}\right) = \ln\left(4^{x+2}\right) \quad \Rightarrow \quad (2x-1)\ln(3) = (x+2)\ln(4) \quad \Rightarrow$$

 $$2\ln(3)x - \ln(3) = \ln(4)x + 2\ln(4) \quad \Rightarrow \quad (2\ln(3) - \ln(4))x = 2\ln(4) + \ln(3) \quad \Rightarrow$$

 $$x = \frac{2\ln(4) + \ln(3)}{2\ln(3) - \ln(4)} \approx 4.7738$$

 (d) $\dfrac{4e^{2x}}{2 + \mathrm{e}^{2x}} = 1$

 $$4\mathrm{e}^{2x} = 2 + \mathrm{e}^{2x} \quad \Rightarrow \quad 3\mathrm{e}^{2x} = 2 \quad \Rightarrow \quad \mathrm{e}^{2x} = \frac{2}{3} \quad \Rightarrow \quad x = \frac{1}{2}\ln\left(\frac{2}{3}\right)$$

 (e) $\ln(8x-7) - \ln(3+4x) = \ln(9/11)$

 $$\ln\left(\frac{8x-7}{3+4x}\right) = \ln\left(\frac{9}{11}\right) \quad \Rightarrow \quad \frac{8x-7}{3+4x} = \frac{9}{11} \quad \Rightarrow \quad 11(8x-7) = 9(3+4x) \quad \Rightarrow$$

 $$52x = 104 \quad \Rightarrow \quad x = 2$$

15. Differentiate:

 (a) $y = x^2 + 4 - \ln(3x)$ $\qquad y' = 2x - \dfrac{1}{3x} \cdot 3 = 2x - \dfrac{1}{x}$

 (b) $y = 5^x - \log_3(x)$ $\qquad y' = 5^x \ln(5) - \dfrac{1}{x\ln(3)}$

(c) $y = 2^{3^x}$ $\qquad y' = 2^{3^x}\ln(2)\cdot 3^x\ln(3)$

(d) $y = 2^{\log_4(x)}$ $\qquad y' = 2^{\log_4(x)}\ln(2)\cdot \dfrac{1}{x\ln(4)}$

(e) $y = 7^{\tan(x)} - \log_2(x^2 - 3x + 5)$

$$y' = 7^{\tan(x)}\ln(7)\cdot\sec^2(x) - \frac{1}{(x^2-3x+5)\ln(2)}\cdot(2x-3)$$

(f) $y = 3^{\arcsin(x)}$ $\qquad y' = 3^{\arcsin(x)}\ln(3)\cdot\dfrac{1}{\sqrt{1-x^2}}$

(g) $y = \log_7(x^2-3) + \frac{\log_4(x)}{1+x^2}$

$$y' = \frac{1}{(x^2-3)\ln(7)}\cdot 2x + \frac{\frac{1}{x\ln(4)}(1+x^2) - \log_4(x)(2x)}{(1+x^2)^2}$$

(h) $y = x^{3x}$. Here we need to rewrite: $x^{3x} = \mathrm{e}^{\ln(x^3 x)} = \mathrm{e}^{3x\ln(x)}$. Now,

$$y' = \mathrm{e}^{3x\ln(x)}\cdot\left(3\ln(x) + 3x\frac{1}{x}\right) = x^{3x}(3\ln(x)+3)$$

(i) $y = \pi^{e^{2^x}}$

$$y' = \pi^{e^{2^x}}\ln(\pi)\cdot\mathrm{e}^{2^{\mathrm{x}}}\cdot 2^x\ln(2)$$

(j) $y = (1+x^2)^x$ Rewrite $(1+x^2)^x = \mathrm{e}^{x\ln(1+x^2)}$.

$$y' = \mathrm{e}^{x\ln(1+x^2)}\cdot\left(\ln(1+x^2) + x\,\frac{1}{1+x^2}\cdot 2x\right) = (1+x^2)^x\left(\ln(1+x^2) + \frac{2x^2}{1+x^2}\right)$$