Homework Questions to replace 1.30-1.31
SOLUTIONS

1. Linearize the given function at the given point:

   To "linearize" means to approximate the function by its tangent line. To compute the tangent line, we need a point and a slope. In general, this is:

$$y - f(a) = f'(a)(x - a) \quad \text{or } y = f(a) + f'(a)(x - a)$$

   (a) $f(x) = \frac{1}{x}$, $a = 1$
   The point: $(1, 1/1) = (1, 1)$. The slope is the derivative evaluated at 1,

$$f'(x) = -\frac{1}{x^2}, \quad f'(1) = -1$$

   so the linearization of $f(x) = \frac{1}{x}$ at $a = 1$ is

$$y - 1 = -(x - 1) \qquad y = 2 - x$$

   (b) $f(x) = \tan^{-1}(x)$, $a = 0$
   The inverse tangent of 0 is 0 (since $\tan(0) = 0$), so the point is the origin, $(0, 0)$. The slope is:

$$\frac{1}{1 + x^2}$$

   evaluated at 1 which gives a slope of 1. Therefore, the linearization is just $y = x$.

   (c) $f(x) = \sqrt{x}$, $a = 9$
   In this case, the point is $(9, 3)$ and the slope is $\frac{1}{2\sqrt{9}} = \frac{1}{6}$. The linearization is

$$y = 3 + \frac{1}{6}(x - 3)$$

2. Approximate $\sqrt{220}$ using a linearization (in this case, you need to choose an appropriate base point, $x = a$)

   We choose a to be a perfect square close to 220. In this case, take $a = 15^2 = 226$. Then we are approximating $f(x) = \sqrt{x}$ close to $a = 226$:

$$y - 15 = \frac{1}{30}(x - 226)$$

   so an approximation to $\sqrt{220}$ is:

$$\sqrt{220} \approx 15 - \frac{1}{30} \cdot 6 = \frac{74}{5}$$

   We note that the approximation is 14.8, while the actual value is approximately 14.832397

3. Suppose that $f(2) = -2$ and $f'(2) = 3$. Give an approximation to $f(2.3)$.

$$f(2.3) \approx f(2) + f'(2)(x-2) = -2 + 3 \cdot (2.3 - 2) = -2 + 3 \cdot 0.3 = -1.1$$

4. The volume of a sphere is dependent on its radius,

$$V = \frac{4}{3}\pi r^3$$

Linearize the volume at $r = 1$, and use it to approximate the volume when $r = 1.1$.

The linearization of volume is the tangent line at $\left(1, \frac{4}{\pi}3\right)$ with slope:

$$4\pi r^2 \text{ at } r = 1 \qquad \rightarrow 4\pi$$

which is:

$$y = \frac{4\pi}{3} + 4\pi(r-1)$$

Compare this with the actual volume if $r = 1.1$.

Compare the actual and estimated volume if $r = 1.2$, and $r = 1.5$. Are the approximations getting better or worse?

Now we compute the approximate and actual volumes using different radii, as suggested:

Radius	Estimated Vol	Actual Vol
1.0	$\frac{4}{3}$	$\frac{4}{3}$
1.1	5.4454	5.5753
1.2	6.7020	7.2382
1.5	10.472	14.1372

The approximation is getting much worse as we go away from $r = 1$.

5. Let $f(x) = x^2 - 3$. Find the $x-$intercept of the tangent line through the graph of f at $x = 4$.

The tangent line at $x = 4$ is given by:

$$y - 13 = 8(x-4)$$

so the $x-$intercept is given by setting $y = 0$ and solving for x,

$$-13 = 8(x-4) \Rightarrow x = 4 - \frac{13}{8} \approx 3.5493$$

6. Let $f(x) = 2\sin(x) - x$. Find the $x-$intercept of the tangent line through the graph of f at $x = \frac{\pi}{2}$.

The tangent line at $x = \frac{\pi}{2}$ is given by:

$$y - \left(2 - \frac{\pi}{2}\right) = -1\left(x - \frac{\pi}{2}\right)$$

Set $y = 0$ and solve for x to get that $x = 2$.

7. Use Newton's Method to find the solution to the equation that is accurate to 5 decimal places:

SOLUTION: We'll write down the solution in column format. Remember the general formula is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(a) $2\sin(x) = x$, $x > 0$ In this case, use $f(x) = 2\sin(x) - x$. Then we note that $x = 0$ is one solution, so we'll use a different guess. Let's start the approximation at $x = \frac{\pi}{2}$, since this was part of a previous exercise. Then:

x	$f(x)$	$f'(x)$	New x
1.570796327	0.429203673	−1.000000000	2.000000000
2.000000000	−0.181405146	−1.832293673	1.900995594
1.900995594	−0.009040087	−1.648463075	1.895511645
1.895511645	−0.000028467	−1.638077989	1.895494267
1.895494267	−0.000000000	−1.638045049	1.895494267
1.895494267			

(b) $\ln(x) = x - 4$, $x > 4$

For the function, we'll take $\ln(x) - x + 4$, and for an initial guess, we'll take $x = 1$. Whoops! In that case, the derivative is zero... Take another initial guess. Maybe $x = 2$. In this case,

x	$f(x)$	$f'(x)$	New x
2.000000000	2.693147181	−0.500000000	7.386294361
7.386294361	−1.386668192	−0.864614115	5.782494275
5.782494275	−0.027659150	−0.827064247	5.749051710
5.749051710	−0.000016789	−0.826058270	5.749031386
5.749031386	−0.000000000	−0.826057655	5.749031386
5.749031386			

(c) $x^3 = 8 - 4x$

In this case, we take $f(x) = x^3 + 4x - 8$, and guess that $x = 2$. Then:

x	$f(x)$	$f'(x)$	New x
2.000000000	8.000000000	16.000000000	1.500000000
1.500000000	1.375000000	10.750000000	1.372093023
1.372093023	0.071528293	9.647917793	1.364679165
1.364679165	0.000225845	9.587047672	1.364655608
1.364655608	0.000000002	9.586854784	1.364655608
1.364655608			