More on Logarithms

Before we discuss why the rules of logarithms work the way they do, there is one fundamental rule we need to know. That is, the following two statements are *equivalent*:

$$\log_a(c) = b \qquad a^b = c$$

That is, a logarithm outputs an exponent.

- Rule: $\log_c(ab) = \log_c(a) + \log_c(b)$

  Proof: We can write $a = c^x$ and $b = c^y$ for some x, y. Using these values, we can also write the equivalent statements that:

  $$a = c^x \Leftrightarrow \log_c(a) = x \quad b = c^y \Leftrightarrow \log_c(b) = y$$

  These are all the pieces we need:

  $$\log_c(ab) = \log_c(c^x c^y) = \log_c(c^{x+y}) = x + y = \log_c(a) + \log_c(b)$$

- Rule: $\log_c(a^b) = b\log_c(a)$

  Proof: Use the same definitions as the first proof, and:

  $$a^b = (c^x)^b = c^{xb}$$

  Now,

  $$\log_c(a^b) = \log_c(c^{xb}) = xb = bx = b\log_c(a)$$

Extra Practice Problems:

1. Show that: $\log_c(a/b) = \log_c(a) - \log_c(b)$
2. Write the following in their equivalent form (without logs)

   $$\log_3(9) = 2 \qquad \log_2(8) = 3 \qquad \log_a(b) = c$$

3. Write the following as logarithms:

   $$4^2 = 16 \qquad 7^3 = 343 \qquad x^y = s$$

4. To compute a logarithm on a calculator, you must use either base 10 or base e. How would you compute $\log_3(5)$?

   $$\text{Let } \log_3(5) = x \Rightarrow 3^x = 5 \Rightarrow \log_a(3^x) = \log_a(5) \Rightarrow x = \frac{\log_a(5)}{\log_a(3)}$$

   where a is any base (greater than 0). Verify this formula by computing $\log_3(5)$ using base 10 and base e- you should get the same answer.

   Note that we have shown that, for any base a,

   $$\log_c(b) = \frac{\log_a(b)}{\log_a(c)}$$

   Therefore, we can work in any base we like- In computer science, people work in base 2. In engineering, base 10, and in Calculus, we work in base e.