M125
Fall 2003

Quiz 4 SOLUTIONS

Show all your work!

1. Find the maximum of $f(x) = 4x + \frac{9}{x}$, for $1 \le x \le 3$.

   First we find the critical points of f, which are where the derivative is zero or does not exist:

$$f'(x) = 4 - \frac{9}{x^2}$$

   The critical points are $x = 0$ (outside the interval) and:

$$4 = \frac{9}{x^2} \Rightarrow 4x^2 = 9 \Rightarrow x^2 = \frac{9}{4} \Rightarrow x = \pm\frac{3}{2}$$

   of which only $x = \frac{3}{2}$ is inside the interval.

   Now we have to check the value of f at the endpoints and critical points.

   At $x = 1$, $f(1) = 4 + \frac{9}{1} = 13$

   At $x = \frac{3}{2}$, $f(3/2) = 4 \cdot \frac{3}{2} + \frac{9}{3/2} = 2 \cdot 3 + 9 \cdot \frac{2}{3} = 6 + 6 = 12$

   At $x = 3$, $f(3) = 4 \cdot 3 + \frac{9}{3} = 12 + 3 = 15$

   Therefore, the maximum value of f is 15 at $x = 3$.

2. Find the maximum of ab^2 if a, b must be positive and $a + b = 5$.

   We have a choice between substitution for a or b. It's easier to substitute for a:

$$a = 5 - b$$

   So $f(b) = (5 - b)b^2 = 5b^2 - b^3$. Now find the critical points:

$$f'(b) = 0 \Rightarrow 10b - 3b^2 = 0 \Rightarrow b(10 - 3b) = 0$$

   so that $b = 0$ or $b = 10/3$. The values of b can range between 0 and 5, so we can check these values as well; $f(0) = f(5) = 0$.

$$f(10/3) = \frac{5}{3} \cdot \frac{100}{9} = \frac{500}{27}$$

3. Extra Credit: In the last problem, what is the answer if 5 is replaced by an arbitrary positive constant k?

   In this case,

$$f(b) = (k - b)b^2 = kb^2 - b^3$$

   so the critical points are:

$$f'(b) = 2kb - 3b^2 = b(2k - 3b) = 0$$

   so $b = 0$ or $b = \frac{2k}{3}$. In the first case, the product is zero, and if $b = k$, the area is also zero. Therefore, if $f(2k/3) > 0$, it will be the max:

$$f(2k/3) = \left(k - \frac{2k}{3}\right) \cdot \left(\frac{2k}{3}\right)^2 = \frac{k}{3} \cdot \frac{4k^2}{9} = \frac{4k^3}{27}$$

   which is positive since $k > 0$.