Worked Problems: Exponentials and Logs (SOLUTIONS)

1. Rewrite the following to its equivalent form. That is, given an exponential expression, rewrite to a logarithmic expression; and given a logarithmic expression, rewrite to an exponential form.

   For each of these, we'll use the equivalence:

   $$\log_a(b) = c \quad \Leftrightarrow \quad a^c = b$$

   For example, since $3^2 = 9$, then $\log_3(9) = 2$.

   (a) $\log_2(x) = 3$
   Solution: $2^3 = x$

   (b) $\mathrm{e}^x\mathrm{e}^5 = 4$
   Solution 1: $\mathrm{e}^x = 4\mathrm{e}^{-5}$, so $\ln(4\mathrm{e}^{-5}) = x$
   We can simplify this using $\log(ab) = \log(a) + \log(b)$ to:

   $$x = \ln(4) + \ln(\mathrm{e}^{-5}) = \ln(4) - 5$$

   Solution 2: From the rule $a^b a^c = a^{b+c}$, we can rewrite the left hand side:

   $$\mathrm{e}^{x+5} = 4 \quad \Rightarrow \quad \ln(4) = x + 5$$

   (c) $2\ln(x) + \ln(x-1) = 4$ Use two rules, $\log(a^b) = b\log(a)$ and $\log(ab) = \log(a) + \log(b)$ to simplify first:

   $$\ln(x^2(x-1)) = 4 \quad \Rightarrow \quad x^2(x-1) = \mathrm{e}^4$$

   (d) $3^{x-5}2^x = 5$ HINT- You might need to use the following: Any number $a > 0$ can be written as $\mathrm{e}^{\ln(a)}$ and $a^b a^c = a^{b+c}$
   Using the hint, we'll rewrite everything in base e: $3 = \mathrm{e}^{\ln(3)}$ and $2 = \mathrm{e}^{\ln(2)}$, so that

   $$3^{x-5}2^x = \mathrm{e}^{\ln(3)\cdot(x-5)}\mathrm{e}^{\ln(2)\cdot x} = \mathrm{e}^{\ln(3)\cdot(x-5)+\ln(2)\cdot x}$$

   Now,

   $$\mathrm{e}^{\ln(3)\cdot(x-5)+\ln(2)\cdot x} = 5 \quad \Rightarrow \quad \ln(3)(x-5) + \ln(2)x = \ln(5)$$

   (e) $\ln(3x) - 4\ln(x+2) = 6$ Use the rules that $\log(a^b) = b\log(a)$ and $\log(a/b) = \log(a) - \log(b)$:

   $$\ln\left(\frac{3x}{(x+2)^4}\right) = 6 \quad \Rightarrow \quad \mathrm{e}^6 = \frac{3x}{(x+2)^4}$$

2. Solve the following for x:

(a) $5^{2x-3} = 4$

$$\log_5(4) = 2x - 3 \quad \Rightarrow \quad x = \frac{\log_5(4) + 3}{2}$$

(b) $3^{x(x-1)} = 2$

$$\log_3(2) = x(x-1) \quad \Rightarrow \quad x^2 - x - \log_3(2) = 0$$

Use the quadratic formula,

$$x = \frac{1 \pm \sqrt{1 + 4\log_3(2)}}{2}$$

(c) $\log_2(x+3) - \log_2(x) = 1$
Rewrite as a single log, remove the logarithms, then solve:

$$\log_2(x+3) - \log_2(x) = \log_2((x+3)/x)$$

Now,

$$\log_2((x+3)/x) = 1 \quad \Rightarrow \quad 2 = \frac{x+3}{x} \quad \Rightarrow \quad 2x = x + 3 \quad \Rightarrow \quad x = 3$$

(d) $2\log_9\left(\frac{x}{3}\right) = 1$
Solution 1:

$$\log_9(x/3) = \frac{1}{2} \Rightarrow \frac{x}{3} = 9^{1/2} = \sqrt{9} = 3 \Rightarrow x = 9$$

Solution 2:

$$\log_9(x^2/9) = 1 \Rightarrow \frac{x^2}{9} = 9 \Rightarrow x^2 = 81 \Rightarrow x = \pm 9$$

We note that $x = -9$ is not in the domain, therefore our answer is $x = 9$

(e) $\log_4\left(\frac{1}{2x}\right) = 3$

$$\frac{1}{2x} = 4^3 \Rightarrow 2x = 4^{-3} \Rightarrow x = \frac{1}{2 \cdot 4^3} = \frac{1}{128}$$

(f) $12^{1/(x-1)} = 4$

$$\log_{12}(4) = \frac{1}{x-1} \Rightarrow x - 1 = \frac{1}{\log_{12}(4)} \Rightarrow x = \frac{1}{\log_{12}(4)} + 1$$

Miscellaneous questions:

1. Determine the domain of $f(x) = \ln(x(x-1))$. For which x can this be rewritten as $\ln(x) + \ln(x-1)$?

   SOLUTION: Use a sign chart, since we need $x(x-1) > 0$:

x	$-$	$+$	$+$
$x-1$	$-$	$-$	$+$
	$x < 0$	$0 < x < 1$	$x > 1$

   From the chart, $x(x-1) > 0$ if $x < 0$ or if $x > 1$.

   In order for $\ln(x(x-1)) = \ln(x) + \ln(x-1)$, both x and $x-1$ must be positive, so this would change the domain to $x > 1$.

2. If $f(x) = 2^x$ and $g(x) = log_2(x)$, compute $f(g(x))$ and $g(f(x))$, and find the domain of each.

   $$f(g(x)) = 2^{\log_2(x)} = x, \text{ for all } x > 0$$
   $$g(f(x)) = \log_2(2^x) = x\log_2(2) = x \text{ for all } x$$

3. If $f(x) = 3^x$, what is $f^{-1}(x)$?

   The inverse of a^x is $\log_a(x)$, so in this case, $f^{-1} = \log_3(x)$.

4. Rewrite 4 as e^A for an appropriate A.

   $$4 = e^{\ln(4)}$$