Continuity and Differentiability Worksheet

(Be sure that you can also do the graphical exercises from the text- These were not included below! Typical problems are like problems 1-3, p. 161; 1-13, p. 171; 33-34, p. 172; 1-4, p. 131; 41, 46-48, 51 p. 176)

1. Definition: A function f is said to be continuous at $x = a$ if: $\lim_{x \to a} f(x) = f(a)$

2. The definition of continuity implies that we have three things to check. What are they? (1) $f(a)$ exists (or $f(a)$ is defined), (2) $\lim_{x \to a} f(x)$ exists. (3) The numbers in (1) and (2) are the same.

3. Finish the definition: A function f is said to be continuous on the interval $[a, b]$ if: f is continuous for every point in (a, b), is left continuous at $x = a$ and right continuous at $x = b$.

4. Finish the definition: The derivative of f at $x = a$ is:
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \quad \text{or} \quad \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

5. Finish the definition: A function f is said to be differentiable on the interval (a, b) if: f is differentiable at each x in the interval (a, b).

6. Why is the interval open in the last definition? Because we need to be able to take the limit (from both sides) of each number in (a, b). If we included the point $x = a$ or $x = b$, we'd have to take a "one-sided" derivative.

7. List three interpretations of the derivative of f at $x = a$. (1) Slope of the Tangent Line to f at $x = a$. (2) Velocity (if f is a displacement function over time), (3) Instantaneous rate of change of f.

8. True or False, and give a short reason:
   (a) If a function is differentiable, then it is continuous. This is true (it was a theorem in class).
   (b) If a function is continuous, then it is differentiable. False. Example: $y = |x|$ at $x = 0$.
   (c) If f is continuous on $[-1, 1]$ and $f(-1) = 4$ and $f(1) = 3$, then there is an $x = r$ so that $f(r) = \pi$. True- This is the statement of the Intermediate Value Theorem.
   (d) If f is continuous at 5, and $f(5) = 2$, then the limit as $x \to 2$ of $f(4x^2 - 11)$ must be 2. True. Because f is continuous, $\lim_{x \to a} f(x) = f(a)$.
   (e) All functions are continuous on their domains. Not True- All of our "basic" functions are, but there are functions that are not continuous anywhere (for example, the function that is zero on the rationals and 1 on the irrationals).

9. Where is each function continuous?
   (a) $f(x) = \sqrt{\dfrac{4 - x^2}{1 - x^2}}$ This function will be continuous on its domain because it is constructed from the functions $4 - x^2$, $1 - x^2$ and $\sqrt{x}$. To find the domain, we require that:
$$\frac{(2+x)(2-x)}{(1+x)(1-x)} \geq 0$$
   so we use a table to solve:

$(2+x)$	$-$	$+$	$+$	$+$	$+$
$(2-x)$	$+$	$+$	$+$	$+$	$-$
$(1+x)$	$-$	$-$	$+$	$+$	$+$
$(1-x)$	$+$	$+$	$+$	$-$	$-$
	$x < -2$	$-2 < x < -1$	$-1 < x < 1$	$1 < x < 2$	$x > 2$

   By the table, the domain is:
   $x \leq -2$, or $-1 < x < 1$, or $x \geq 2$. which is also where $f(x)$ is continuous.

(b) $f(x) = \sin^{-1}(1 - x^2)$

First, recall that the domain of $\sin^{-1}(x)$ is $-1 \le x \le 1$. Therefore, the domain of $\sin^{-1}(1-x^2)$ is where $-1 \le 1 - x^2 \le 1$. This implies that $-2 \le -x^2 \le 0$, or where $0 \le x^2 \le 2$.

Thus, the answer is: The function f is continuous on its domain, $-\sqrt{2} \le x \le \sqrt{2}$.

(c) $f(x) = \ln\left(\dfrac{x+3}{x-5}\right)$

This function will be continuous on its domain- The function $\ln(x)$ has a domain: $x > 0$, so $\ln(\frac{x+3}{x-5})$ has a domain that satisfies: $\frac{x+3}{x-5} > 0$. Use a table to solve:

$x+3$	$-$	$+$	$+$
$x-5$	$-$	$-$	$+$
	$x < -3$	$-3 < x < 5$	$x > 5$

Conclusion: The function is continuous for $x < -3$ or $x > 5$.

(d) $f(x) = \dfrac{x}{x^2 + 5x + 6}$

Here, we need to make sure that $x^2 + 5x + 6 \neq 0$, so solve by factoring.

Conclusion: The function is continuous on all reals except where $x = -3$ and $x = -2$.

10. Explain why the function is discontinuous at the given point, $x = a$.

(a) $f(x) = \ln|x+3|$ at $a = -3$ (Extra: Is f continuous everywhere else?)

f is not continuous at $a = -3$ because f is not defined for $a = -3$. (Yes, f is continuous for all other x).

(b)

$$f(x) = \begin{cases} \frac{x^2-2x-8}{x-4}, & \text{if } x \neq 4 \\ 3, & \text{if } x = 4 \end{cases} \qquad a = 4$$

For this function, (1) f is defined at $a = 4$, and $f(4) = 3$. (2) $\lim_{x\to 4} f(x) = \lim_{x\to 4} x + 2 = 6$. (3) The answers for (1) and (2) are not the same, so f is not continuous at $a = 4$.

(c) $f(x) = \frac{x^2-1}{x+1}$, at $a = -1$

For this function, f is not continuous at $a = -1$ because $f(-1)$ is not defined.

(d)

$$f(x) = \begin{cases} 1 - x, & \text{if } x \le 2 \\ x^2 - 2x, & \text{if } x > 2 \end{cases} \qquad a = 2$$

We again check the three properties: (1) $f(2) = -1$, so f is defined at $a = 2$. (2) $\lim_{x\to 2} f(x)$ does not exist- The limit coming from the right is 0, the limit coming from the left is -1. We don't have to check the third property- f is not continuous because the limit at $a = 2$ does not exist.

11. For each function, determine the value of the constant so that f is continuous everywhere:

(a)

$$f(x) = \begin{cases} \frac{x^2-16}{x-4}, & \text{if } x \neq 4 \\ C, & \text{if } x = 4 \end{cases}$$

First, $f(4) = C$, so that does not restrict our choice of C. Next, we want the limit to exist (and be equal to C), so: $C = \lim_{x\to 4} \dfrac{x^2 - 16}{x - 4} = 8$.

(b)

$$f(x) = \begin{cases} 3x^2 - 1, & \text{if } x < 0 \\ cx + d, & \text{if } 0 \le x \le 1 \\ \sqrt{x+8}, & \text{if } x > 1 \end{cases}$$

First, the only values of x to consider are $x = 0$ and $x = 1$. In these cases,

$$f(0) = d \text{ and } f(1) = c + d$$

so these exist for all values of c, d.
Next make sure the limits match at $x = 0$ and at $x = 1$ as we come in from the right and left.
At $x = 0$:

$$\lim_{x \to 0^+} f(x) = d \text{ and } \lim_{x \to 0^-} f(x) = -1$$

So $d = -1$. Using this, we check $x = 1$:

$$\lim_{x \to 1^+} f(x) = 3 \text{ and } \lim_{x \to 1^-} f(x) = c - 1$$

so $c = 4$.

(c)

$$f(x) = \begin{cases} \frac{\sqrt{7x+2} - \sqrt{6x+4}}{x-2}, & \text{if } x \ge -\frac{2}{7}, \text{ and } x \ne 2 \\ k, & \text{if } x = 2 \end{cases}$$

We want the limit to exist, and the value of the function at $x = 2$ should be equal to that limit.
First, the limit as $x \to 2$:

$$\lim_{x \to 2} \frac{\sqrt{7x+2} - \sqrt{6x+4}}{x-2} = \lim_{x \to 2} \frac{\sqrt{7x+2} - \sqrt{6x+4}}{x-2} \cdot \frac{\sqrt{7x+2} + \sqrt{6x+4}}{\sqrt{7x+2} + \sqrt{6x+4}} =$$

$$\lim_{x \to 2} \frac{1}{\sqrt{7x+2} + \sqrt{6x+4}} = \frac{1}{8}$$

The value of f at $x = 2$ is k. For this to match the limit, $k = \frac{1}{8}$

12. If f and g are continuous functions with $f(3) = 4$ and $\lim_{x \to 3}[2f(x) - g(x)] = 5$, what is $g(3)$?

    By continuity,

$$\lim_{x \to 3}[2f(x) - g(x)] = 2 \lim_{x \to 3} f(x) - \lim_{x \to 3} g(x) = 2f(3) - g(3) = 8 - g(3)$$

so that we now have:

$$8 - g(3) = 5$$

so $g(3) = 3$.

13. Show that there must be at least one real solution to $x^5 - x^2 - 4 = 0$. This is an Intermediate Value Theorem, where 0 is the Intermediate Value. Therefore, we need to find an x so that $x^5 - x^2 - 4 < 0$ and an x so that $x^5 - x^2 - 4 > 0$. For example, if $x = 1$, then we get a -4. If $x = 2$, we get $32 - 4 - 4 = 24 > 0$. Therefore, a solution to the equation is somewhere between $x = 1$ and $x = 2$.

14. Each limit is the derivative of some function at some number a. State f and a in each case:

    (a) $\lim_{h \to 0} \frac{\sqrt{1+h} - 1}{h}$
    $f(x) = \sqrt{x}$ at $a = 1$.

(b) $\lim_{x\to 1} \frac{x^9-1}{x-1}$

$f(x) = x^9$ at $a = 1$

(c) $\lim_{t\to 0} \frac{\sin\left(\frac{\pi}{2}+t\right)-1}{t}$

$f(x) = \sin(x)$ at $a = \frac{\pi}{2}$

15. For each function below, compute the derivative using the definition. Also state the domain of the original function, and the domain of the derivative function.

(a) $f(x) = \sqrt{1+2x}$ Domain of f: $x \geq -\frac{1}{2}$

$$\lim_{h\to 0} \frac{\sqrt{1+2(x+h)}-\sqrt{1+2x}}{h} = \lim_{h\to 0} \frac{\sqrt{1+2(x+h)}-\sqrt{1+2x}}{h} \cdot \frac{\sqrt{1+2(x+h)}+\sqrt{1+2x}}{\sqrt{1+2(x+h)}+\sqrt{1+2x}} =$$

$$\lim_{h\to 0} \frac{1+2x+2h-1-2x}{h(\sqrt{1+2(x+h)}+\sqrt{1+2x})} = \lim_{h\to 0} \frac{2}{(\sqrt{1+2(x+h)}+\sqrt{1+2x})} = \frac{1}{\sqrt{1+2x}}$$

(b) $g(x) = \frac{1}{x^2}$

$$\lim_{h\to 0} \frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h} = \lim_{h\to 0} \frac{\frac{x^2-(x+h)^2}{x^2(x+h)^2}}{h} = \lim_{h\to 0} \frac{-2x-h}{x^2(x+h)^2} = -\frac{2}{x^3}$$

(c) $h(x) = x + \sqrt{x}$

$$\lim_{h\to 0} \frac{[x+h+\sqrt{x+h}]-[x+\sqrt{x}]}{h} = \lim_{h\to 0} \frac{h+\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0} \left(1+\frac{\sqrt{x+h}-\sqrt{x}}{h}\right) =$$

$$1+\lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = 1+\lim_{h\to 0} \frac{h}{h(\sqrt{x+h}+\sqrt{x})} = 1+\frac{1}{2\sqrt{x}}$$

(d) $f(x) = \frac{2}{\sqrt{3-x}}$

$$\lim_{h\to 0} \frac{\frac{2}{\sqrt{3-(x+h)}}-\frac{2}{\sqrt{3-x}}}{h} = \lim_{h\to 0} \frac{2(\sqrt{3-x}-\sqrt{3-x-h})}{h\sqrt{3-x}\sqrt{3-x-h}} \cdot \frac{\sqrt{3-x}+\sqrt{3-x-h}}{\sqrt{3-x}+\sqrt{3-x-h}} =$$

$$\lim_{h\to 0} \frac{2}{\sqrt{3-x}\sqrt{3-x-h}(\sqrt{3-x}+\sqrt{3-x-h})} = \frac{2}{(3-x)2\sqrt{3-x})} = \frac{1}{(3-x)^{3/2}}$$

(e) $f(x) = \frac{x}{x^2-1}$

$$\lim_{h\to 0} \frac{\frac{x+h}{(x+h)^2-1}-\frac{x}{x^2-1}}{h} = \lim_{h\to 0} \frac{(x+h)(x^2-1)-x((x+h)^2-1)}{h((x+h)^2-1)(x^2-1)} = \lim_{h\to 0} \frac{-hx^2+xh^2-h}{h((x+h)^2-1)(x^2-1)} =$$

$$\frac{-x^2-1}{(x^2-1)^2}$$

16. Let $f(x) = \sqrt{x}$.

(a) Use $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ to compute $f'(a)$, for $a \neq 0$.

$$\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} = \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})} = \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}} = \frac{1}{2\sqrt{x}}$$

ALTERNATIVE: You could also multiply numerator and denominator by $\sqrt{x}+\sqrt{a}$ and get the same result.

(b) Show that $f'(0)$ does not exist. What does this mean with respect to the graph of f at $a = 0$?

From our formula, we see that, as $x \to 0$, $f'(x) \to \infty$, which means that at $x = 0$, there is a vertical tangent line.

17. Given f below, where is f not continuous?

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ 5 - x, & \text{if } 0 < x < 4 \\ \frac{1}{5-x}, & \text{if } x \geq 4 \end{cases}$$

Not continuous at: $x = 0$, because the limit from the right is not the limit from the left. It is continuous at $x = 4$, and for all other x.

18. Let $f(x) = x^3 - 2x$. (a) Find $f'(2)$. (b) Compute the equation of the line tangent to f at the point $(2, 4)$.

$$\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{(2+h)^3 - 2(2+h) - (2^3 - 2(2))}{h} = \lim_{h\to 0} \frac{h(10 + 6h + h^2)}{h} = 10$$

The equation of the line through $(2, 4)$ with slope 10 is

$$y - 4 = 10(x - 2)$$

which is the tangent line.

19. Sketch the graph of a function that satisfies the following conditions: $g(0) = 0$, $g'(0) = 3$, $g'(1) = 0$, $g'(2) = 1$

Your graph should have at least: A point at $(0, 0)$ with the curve going through the origin fairly steeply (local slope of 3), where the curve goes through $x = 1$, the curve should be flat (slope of zero), and finally, where the curve goes through $x = 2$, the slope should be about 1.

20. Find the slope of the line tangent to $y = x^2 + 2x$ at $x = -3$, then compute the equation of the line.

$$m_{\tan} = \lim_{h\to 0} \frac{f(-3+h) - f(-3)}{h} = \lim_{h\to 0} \frac{(-3+h)^2 + 2(-3+h) - 3}{h} = \lim_{h\to 0} \frac{-4h + h^2}{h} = -4$$

The tangent line has the equation: (In general: $y - f(a) = f'(a)(x - a)$)

$$y - 3 = -4(x + 3)$$