Review Questions, Exam 3 (UPDATE)

1. A child is flying a kite. If the kite is 90 feet above the child's hand level and the wind is blowing it on a horizontal course at 5 feet per second, how fast is the child paying out cord when 150 feet of cord is out? (Assume that the cord forms a line- actually an unrealistic assumption).

   Note: As with all word problems, your notation may be different than mine First, draw a picture of a right triangle, with height 90, hypotenuse $y(t)$, the other leg $x(t)$. Note that these are varying until after we differentiate. So, we have that

   $$(y(t))^2 = (x(t))^2 + 90^2$$

   and

   $$2y(t)\frac{dy}{dt} = 2x(t)\frac{dx}{dt}$$

   We want to find $\frac{dy}{dt}$ when $y = 150$ and $\frac{dx}{dt} = 5$. We need to know $x(t)$, so use the first equation:

   $$150^2 - 90^2 = x^2 \Rightarrow x = 120$$

   so

   $$2 \cdot 150\frac{dy}{dt} = 2 \cdot 120 \cdot 5 \Rightarrow \frac{dy}{dt} = 4$$

2. Use differentials to approximate the increase in area of a soap bubble, when its radius increases from 3 inches to 3.025 inches ($A = 4\pi r^2$)

   $$dA = 8\pi r\, dr$$

   and $r = 3$, $dr = 0.025$, so

   $$dA = 8 \cdot 3 \cdot 0.025 \cdot \pi = 0.6\pi$$

3. True or False, and give a short reason:

   (a) If air is being pumped into a spherical rubber balloon at a constant rate, then the radius will increase, but at a slower and slower rate.
   True. $V = \frac{4}{3}\pi r^3$, so

   $$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

   Since $\frac{dV}{dt}$ is constant, lets call it k. Solving for $\frac{dr}{dt}$, we get:

   $$\frac{dr}{dt} = \frac{k}{4\pi r^2}$$

   so as r gets large, $\frac{dr}{dt}$ gets smaller (but does stay positive).

   (b) FALSE. To see if $f(x)g(x)$ is increasing or decreasing, we need to take its derivative:

   $$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

   From what we're told, $f'(x) > 0$ and $g'(x) > 0$. But, the derivative also involves $f(x)$ and $g(x)$. Since we don't know the signs of f, g we cannot say that $f(x)g(x)$ is increasing or decreasing.
   SIDE NOTE: If this were $f(x) + g(x)$ instead of $f(x)g(x)$, we could say that $f + g$ is increasing, since $(f + g)' = f'(x) + g'(x) > 0$

   (c) If $y = x^5$, then $dy \geq 0$
   Before answering, look at dy:

   $$dy = x^4\, dx$$

   So, as long as dx is positive, $dy \geq 0$. However, dx (or Δx) can be negative, which would make dy negative. So, the answer is False.

(d) If a car *averages* 60 miles per hour over an interval of time, then at some instant, the speedometer must have read exactly 60.

True. This is exactly the Mean Value Theorem. The velocity (call it $f'(c)$) at some point in the time interval must have been equal to the average velocity between times a and b,

$$\frac{f(b) - f(a)}{b - a}$$

(we can assume the position function is differentiable).

(e) A global maximum is always a local maximum. False, since endpoints cannot be local maximums (by definition, we ruled those out).

(f) The linear function $f(x) = ax + b$, where a, b are constant, and $a \neq 0$, has no minimum value on any open interval. (An interval is open if it does not include its endpoints).

True. If $a \neq 0$, then $f(x)$ is not a horizontal line, which means it must go up or down in the interval- so *if* the endpoints were included, then one of the endpoints would be the min (or max). But since the endpoints are NOT included, the function will never actually “hit” its minimum.

(g) Suppose P and Q are two points on the surface of the sea, with Q lying generally to the east of P. It is possible to sail from P to Q (always sailing roughly east), without *ever* sailing in the exact direction from P to Q.

This is meant to get you to think about the Mean Value Theorem, and it is false. Think about the path the sailboat takes as a function $f(t)$ (draw a picture, with $(0, f(0)) = P$ and $(1, f(1)) = Q$, for example). The direction from P to Q is the slope of the line, and the mean value theorem says that at some point in between, the slope of the tangent line must also point in that direction.

(h) I f$f(x) = 0$ has three distinct solutions, then $f'(x) = 0$ must have (at least) two solutions. TRUE- By the Mean Value Theorem, if a, b, c are the three solutions to $f(x) = 0$, then there are values c_1, c_2 so that:

$$f'(c_1) = \frac{f(b) - f(a)}{b - a} = 0 \qquad f'(c_2) = \frac{f(c) - f(b)}{c - b} = 0$$

Similarly, $f''(x) = 0$ must have one solution in the interval (c_1, c_2).

4. Show that, if $f(x)$ is increasing, then $1/f(x)$ is decreasing.

If $f(x)$ is increasing, then $f'(x) > 0$. Take the derivative of $1/f(x)$ to show that it is negative:

$$\frac{d}{dx}(1/f(x)) = -(f(x))^{-2} f'(x) = -\frac{f'(x)}{(f(x))^2}$$

The denominator is always positive, $f'(x) > 0$, so this quantity is always negative. That shows that $1/f(x)$ is decreasing if $f(x)$ is increasing.

5. The first derivative test is a test for local max/min. That is, if the first derivative changes sign from positive to negative, we have found a LOCAL MAX. If the sign change is from negative to positive, we have found a LOCAL MIN.

The second derivative test is also a test for local max/min. If $f'(c) = 0$ and $f''(c) > 0$, then a local maximum occurs at $x = c$. If $f'(c) = 0$ and $f''(c) < 0$, then a local minimum occurs at $x = c$. Finally, if $f'(c) = 0$ and $f''(c) = 0$, then we cannot conclude anything.

6. The three values theorems are:

(a) The Intermediate Value Theorem: If f is cont. on $[a, b]$, and N is any value between $f(a)$ and $f(b)$, then there is a c in $[a, b]$ where $f(c) = N$.

(b) The Extreme Value Theorem: If f is continuous on $[a, b]$, then f attains both a global max and a global min on $[a, b]$.

(c) The Mean Value Theorem: If f is continuous on $[a, b]$ and f is differentiable on (a, b), then there is a c in (a, b) such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

7. If $f(x) = 6 - x^2$, $x = -2$, $\Delta x = -1$, then:

$$\Delta y = f(x + \Delta x) - f(x) = f(-3) - f(-2) = -3 - 2 = -5$$

$$dy = -2x\, dx = -2(-2)(-1) = -4$$

We'll show the graph in class.

8. Find the equation of the tangent line to

$$y = \sqrt{x} \mathrm{e}^{x^2} (x^2 + 1)^{10}$$

at $x = 0$. First, the point is $(0, 0)$. Next, the slope is found by differentiating. In this case (for review), you can use logarithmic differentiation:

First take logs and simplify:

$$\ln(y) = \ln(x^{1/2}) + \ln(e^{x^2}) + \ln((x^2 + 1)^{10}) = \frac{1}{2}\ln(x) + x^2 + 10\ln(x^2 + 1)$$

Differentiate:

$$\frac{1}{y} y' = \frac{1}{2x} + 2x + \frac{20x}{x^2 + 1}$$

And the last step is to multiply both sides by y, then substitute in for y:

$$y' = \sqrt{x} \mathrm{e}^{x^2} (x^2 + 1)^{10} \left(\frac{1}{2x} + 2x + \frac{20x}{x^2 + 1} \right)$$

Thus, $y' = 0$ and $y = 0$ is the tangent line.

9. Estimate by linear approximation the change in the indicated quantity.

   (a) The volume, $V = s^3$ of a cube, if its side length s is increased from 5 inches to 5.1 inches.

   $$dV = 3s^2\, ds \Rightarrow dV = 3 \cdot 25 \cdot 0.1 = 7.5$$

   (b) The volume, $V = \frac{4}{3}\pi r^3$ of a sphere, if the radius changes from 2 to 2.1:

   $$dV = 4\pi r^2 \cdot dr = 4\pi(2)^2 \cdot (0.1) = 1.6\pi$$

   (c) The volume, $V = \frac{1000}{p}$, of a gas, if the pressure p is decreased from 100 to 99.

   $$dV = \frac{-1000}{p^2}\, dp \Rightarrow dV = \frac{-1000}{100^2}(-1) = 0.1$$

   (d) The period of oscillation, $T = 2\pi\sqrt{\frac{L}{32}}$, of a pendulum, if its length L is increased from 2 to 2.2. First, note that:

   $$T = \frac{2\pi}{\sqrt{32}}\sqrt{L}$$

   which makes it slightly easier to differentiate:

   $$dT = \frac{\pi}{\sqrt{32}} L^{-1/2}\, dL \Rightarrow dT = \frac{\pi}{\sqrt{64}} \cdot 0.2 \approx 0.0785$$

10. For the following problems, find where f is increasing or decreasing. If asked, also check concavity.

    (a) $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ (Also check for concave up/down)

    $$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x - 2)(x - 1)$$

    Build a chart:

x	$-$	$+$	$+$	$+$
$(x-2)$	$-$	$-$	$-$	$+$
$(x-1)$	$-$	$-$	$+$	$+$
	$x<0$	$0<x<1$	$1<x<2$	$x>2$

Now, f is increasing on $0<x<1$ and $x>2$, and f is decreasing for $x<0$ and $1<x<2$.
Now, for concavity:

$$f''(x) = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2)$$

This is not factorable into "nice" numbers, but we can use the quadratic formula to see that:

$$3x^2 - 2x - 2 = 0 \Rightarrow x = \frac{1 \pm \sqrt{7}}{3}$$

so that $f''(x)$ might change sign at those points:

$3x^2-2x-2$	$+$	$-$	$+$
	$x < \frac{1-\sqrt{7}}{3}$	$\frac{1-\sqrt{7}}{3} < x < \frac{1+\sqrt{7}}{3}$	$x > \frac{1+\sqrt{7}}{3}$

so f is concave down for $\frac{1-\sqrt{7}}{3} < x < \frac{1+\sqrt{7}}{3}$, concave up everywhere else (except at the roots, where $f''(x)=0$).

(b) $f(x) = \frac{x}{x+1}$

$$f'(x) = \frac{1\cdot(x+1) - 1\cdot x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

so f is always increasing ($f'(x)$ is always positive).
For concavity, we check the second derivative:

$$f''(x) = -2(x+1)^{-3} = \frac{-2}{(x+1)^3}$$

We could build a chart, or, by inspection, we see that if $x<-1$, the denominator is negative, so overall f'' is positive: For $x<-1$, f is concave up.
For $x>-1$, the denominator is positive, so overall f'' is negative: For $x>-1$, f is concave down.

(c) $f(x) = x\sqrt{x^2+1}$

$$f'(x) = \sqrt{x^2+1} + x(\frac{1}{2}(x^2+1)^{-1/2}2x) = \sqrt{x^2+1} + \frac{x^2}{\sqrt{x^2+1}}$$

From this, we see that every term in the expression is positive, so f is always increasing. To check the second derivative, it might be useful to simplify first:

$$f'(x) = \sqrt{x^2+1}\frac{\sqrt{x^2+1}}{\sqrt{x^2+1}} + \frac{x^2}{\sqrt{x^2+1}} = \frac{2x^2+1}{\sqrt{x^2+1}}$$

Now computing the second derivative:

$$f''(x) = \frac{4x\cdot\sqrt{x^2+1} - (4x^2+1)\frac{x}{\sqrt{x^2+1}}}{x^2+1} = \frac{\frac{4x(x^2+1) - x(4x^2+1)}{\sqrt{x^2+1}}}{x^2+1} = \frac{3x}{(x^2+1)^{3/2}}$$

The denominator is always positive, but the numerator changes sign at the origin, so f is concave down if $x<0$, concave up if $x>0$.

11. Show that the given function satisfies the hypotheses of the Mean Value Theorem. Find all numbers c in that interval that satisfy the conclusion of that theorem. For comparison purposes, given these functions and intervals, what would the Intermediate Value Theorem conclude? Finally, find the global max and global min for each function.

(a) $f(x) = x^3$, $[-1, 1]$
Since f is a polynomial, it is continuous and differentiable everywhere. In particular, it will satisfy the MVT on $[-1, 1]$.
First, we see that $f'(c)$ will be $3c^2$, so we need to solve:

$$3c^2 = \frac{f(1) - f(-1)}{1 - (-1)} = 1$$

so $c = \pm\sqrt{\frac{1}{3}} \approx \pm 0.5773$.
The IVT would say that we could find $c \in [-1, 1]$ so that $f(c)$ is any number between $f(1)$ and $f(-1)$ (in this case, any number between 1 and -1).
Checking endpoints $-1, 1$ and the critical point $x = 0$, the global max is 1, the global min is -1.

(b) $f(x) = \sqrt{x - 1}$, $[2, 5]$
The function f will be continuous on its domain, $x > 1$. The derivative is $\frac{1}{2}(x - 1)^{-1/2}$, which exists for $x > 1$. Therefore, f is continuous on $[2, 5]$ and differentiable on $(2, 5)$. Now we apply the MVT:

$$\frac{1}{2}(c - 1)^{-1/2} = \frac{2 - 1}{5 - 2} = \frac{1}{3}$$

so that

$$\frac{1}{\sqrt{c - 1}} = \frac{2}{3} \Rightarrow \sqrt{c - 1} = \frac{3}{2} \Rightarrow c - 1 = \frac{9}{4} \Rightarrow c = \frac{13}{4} = 3.25$$

The IVT would say that there is a $c \in [2, 5]$ so that $f(c)$ is any value between $f(2) = 1$ and $f(5) = 2$.
For the global max/min, check endpoints and the critical point $x = 1$ to find a global max of 2, and a global min of 0.

(c) $f(x) = x + \frac{1}{x}$, $[1, 5]$
First, f is continuous for $x > 0$, and the derivative:

$$f'(x) = 1 - x^{-2}$$

also exists for $x \neq 0$. So the MVT will apply on the interval $[1, 5]$. We compute $f(5) = 5 - \frac{1}{5} = \frac{24}{5}$ and $f(1) = 1 + 1 = 2$, and so:

$$1 - c^{-2} = \frac{\frac{24}{5} - 2}{5 - 1} = \frac{7}{10}$$

so that

$$c^{-2} = \frac{3}{10} \Rightarrow c^2 = \frac{10}{3} \Rightarrow c = \pm\sqrt{\frac{10}{3}} \approx \pm 1.825$$

We choose the value of c in $[1, 5]$, so $c = \sqrt{\frac{10}{3}}$
The IVT would say that there is a c in $[1, 5]$ so that $f(c)$ is any number between 2 and $\frac{24}{5}$.
To find the global max/min, check endpoints and critical points ($x = 1$, which is already an endpoint) to find that the global maximum is 26/5 (at $x = 5$) and the global minimum is 3/2 (at $x = 1$.

12. Show that $f(x) = x^{2/3}$ does not satisfy the hypotheses of the mean value theorem on $[-1, 27]$, but nevertheless, there is a c for which:

$$f'(c) = \frac{f(27) - f(-1)}{27 - (-1)}$$

Find the value of c.

First, $f(x) = x^{2/3}$ is a continuous function, but its derivative is $f'(x) = \frac{2}{3}x^{-1/3}$, which does not exist at $x = 0$.

Let's see if we can find a suitable c anyway:

$$\frac{2}{3}c^{-1/3} = \frac{9 - (-1)}{27 - (-1)} = \frac{10}{28}$$

$$c^{-1/3} = \frac{15}{28} \Rightarrow c^{1/3} = \frac{28}{15} \Rightarrow c = \left(\frac{28}{15}\right)^3 \approx 6.504$$

13. At 1:00 PM, a truck driver picked up a fare card at the entrance of a tollway. At 2:15 PM, the trucker pulled up to a toll booth 100 miles down the road. After computing the trucker's fare, the toll booth operator summoned a highway patrol officer who issued a speeding ticket to the trucker. (The speed limit on the tollway is 65 MPH).

    (a) The trucker claimed that he hadn't been speeding. Is this possible? Explain. Nope. Not possible. The trucker went 100 miles in 1.25 hours, which is not possible if you go (at a maximum) of 65 miles per hour (which would only get you (at a max) 81.25 miles). In terms of the MVT:

    $$\frac{\text{Change in Position}}{\text{Change in time}} = \frac{100}{1.25} = 80$$

    So we can guarantee that at some point in time, the trucker's speedometer read exactly 80 MPH.

    (b) The fine for speeding is \$35.00 plus \$2.00 for each mph by which the speed limit is exceeded. What is the trucker's minimum fine? By the last computation, the trucker had an *average* speed of 80 mph, so we can guarantee (by the MVT) that at some point, the speedometer read exactly 80. So, this gives \$35.00 + \$2.00 (15) = \$65.00

14. Let $f(x) = \frac{1}{x}$

    (a) What does the Extreme Value Theorem (EVT) say about f on the interval $[0.1, 1]$? Since f is continuous on this closed interval, there is a global max and global min (on the interval).

    (b) Although f is continuous on $[1, \infty)$, it has no minimum value on this interval. Why doesn't this contradict the EVT? The EVT was stated on an interval of the form $[a, b]$, which implies that we cannot allow a, b to be infinite.

15. Let f be a function so that $f(0) = 0$ and $\frac{1}{2} \leq f'(x) \leq 1$ for all x. Use the Mean Value Theorem to explain why $f(2)$ cannot be 3.

    We know that:

    $$\frac{f(2) - f(0)}{2 - 0} = f'(c)$$

    for some c in $(0, 2)$. Using the restrictions on the derivative,

    $$\frac{1}{2} \leq \frac{f(2)}{2} \leq 1$$

    so that $1 \leq f(2) \leq 2$.

16. Sketch the graph of a function that satisfies all of the given properties:

    $$f'(-1) = 0, f'(1) \text{ does not exist}, f'(x) < 0 \text{ if } \|x\| < 1, f'(x) > 0 \text{ if } \|x\| > 1$$

    $$f(-1) = 4, f(1) = 0, f''(x) > 0 \text{ if } x > 0$$

    We'll give the answer to this one in class.

17. Find the local maximums and local minimums of f using both the first and second derivative tests:

    $$f(x) = x + \sqrt{1 - x}$$

    First, note that the domain of this function is: $x \leq 1$

    To use the either test, we have to find the zeros to $f'(x)$ first:

    $$f'(x) = 1 + \frac{1}{2}(1 - x)^{-1/2}(-1) = 1 - \frac{1}{2\sqrt{1 - x}}$$

Set this to zero and solve:

$$1 = \frac{1}{2\sqrt{1-x}} \Rightarrow \quad 2\sqrt{1-x} = 1 \Rightarrow \quad 1 - x = \frac{1}{4} \Rightarrow x = \frac{3}{4}$$

Now, if $x < \frac{3}{4}$, then $f'(x)$ will be positive (put in a sample x, say $x = 0$). And if $\frac{3}{4} < x < 1$, $f'(x) < 0$. Therefore, f is increasing for $x < \frac{3}{4}$, and f is decreasing for $\frac{3}{4} < x < 1$. Therefore, there is a local maximum that occurs at $x = \frac{3}{4}$ (This is the first derivative test).

For the second derivative test, compute $f''(x)$, then evaluate at $x = \frac{3}{4}$:

$$f'(x) = 1 - \frac{1}{2}(1-x)^{-1/2} \Rightarrow \quad f''(x) = \frac{1}{4}(1-x)^{-3/2}(-1)$$

so that $f''(3/4) = -2$. Therefore, f is concave down at $x = 3/4$, and by the second derivative test, there is a local max at $x = 3/4$.

(NOTE: In general, we wouldn't need to perform both tests, but in this problem, we were asked to do both).

18. Related Rated Solutions:

To do these problems, you may need to use one or more of the following: The Pythagorean Theorem, Similar Triangles, Proportionality (A is proportional to B means that $A = kB$, for some constant k).

(a) The top of a 25-foot ladder, leaning against a vertical wall, is slipping down the wall at a rate of 1 foot per second. How fast is the bottom of the ladder slipping along the ground when the bottom of the ladder is 7 feet away from the base of the wall?

First, make a sketch of a triangle whose hypotenuse is the ladder. Let $y(t)$ be the height of the ladder with the vertical wall, and let $x(t)$ be the length of the bottom of the ladder with the vertical wall. Then

$$x^2(t) + y^2(t) = 25^2$$

The problem can then be interpreted as: If $\frac{dy}{dt} = -1$, what is $\frac{dx}{dt}$ when $x = 7$?
Differentiating with respect to time:

$$2x(t)\frac{dx}{dt} + 2y(t)\frac{dy}{dt} = 0$$

we have numbers for $\frac{dy}{dt}$ and $x(t)$- we need a number for y in order to solve for $\frac{dx}{dt}$. Use the original equation, and

$$7^2 + y^2(t) = 25^2 \Rightarrow y = 24$$

Now plug everything in and solve for $\frac{dx}{dt}$:

$$2 \cdot 7 \cdot \frac{dx}{dt} + 2 \cdot 24 \cdot (-1) = 0 \Rightarrow \frac{dx}{dt} = \frac{24}{7}$$

(b) A 5-foot girl is walking toward a 20-foot lamppost at a rate of 6 feet per second. How fast is the tip of her shadow (cast by the lamppost) moving?

Let $x(t)$ be the distance of the girl to the base of the post, and let $y(t)$ be the distance of the tip of the shadow to the base of the post. If you've drawn the right setup, you should see similar triangles...

$$\frac{\text{Hgt of post}}{\text{Hgt of girl}} = \frac{\text{Dist of tip of shadow to base}}{\text{Dist of girl to base}}$$

In our setup, this means:

$$\frac{20}{5} = \frac{y(t)}{y(t) - x(t)}$$

With a little simplification, we get:

$$3y(t) = 4x(t)$$

We can now interpret the question as asking what $\frac{dy}{dt}$ is when $\frac{dx}{dt} = -6$. Differentiating, we get

$$3\frac{dy}{dt} = 4\frac{dx}{dt}$$

so that the final answer is $\frac{dy}{dt} = -8$, which we interpret to mean that the tip of the shadow is approaching the post at a rate of 8 feet per second.

(c) Under the same conditions as above, how fast is the length of the girl's shadow changing?

Let $L(t)$ be the length of the shadow at time t. Then, by our previous setup,

$$L(t) = y(t) - x(t)$$

so $\frac{dL}{dt} = \frac{dy}{dt} - \frac{dx}{dt} = -8 - (-6) = -2$.

(d) A rocket is shot vertically upward with an initial velocity of 400 feet per second. Its height s after t seconds is $s = 400t - 16t^2$. How fast is the distance changing from the rocket to an observer on the ground 1800 feet away from the launch site, when the rocket is still rising and is 2400 feet above the ground?

We can form a right triangle, where the launch site is the vertex for the right angle. The height is $s(t)$, given in the problem, the length of the second leg is fixed at 1800 feet. Let $u(t)$ be the length of the hypotenuse. Now we have:

$$u^2(t) = s^2(t) + 1800^2$$

and we can interpret the question as asking what $\frac{du}{dt}$ is when $s(t) = 2400$. Differentiating, we get

$$2u(t)\frac{du}{dt} = 2s(t)\frac{ds}{dt} \text{ or } u(t)\frac{du}{dt} = s(t)\frac{ds}{dt}$$

To solve for $\frac{du}{dt}$, we need to know $s(t)$, $u(t)$ and $\frac{ds}{dt}$. We are given $s(t) = 2400$, so we can get $u(t)$:

$$u(t) = \sqrt{2400^2 - 1800^2} = 3000$$

Now we need $\frac{ds}{dt}$. We are given that $s(t) = 400t - 16t^2$, so $\frac{ds}{dt} = 400 - 32t$. That means we need t. From the equation for $s(t)$,

$$2400 = 400t - 16t^2 \Rightarrow -16t^2 + 400t - 2400 = 0$$

Solve this to get $t = 10$ or $t = 15$. Our rocket is on the way up, so we choose $t = 10$. Finally we can compute $\frac{ds}{dt} = 400 - 32(10) = 80$. Now,

$$3000\frac{du}{dt} = 2400(80)$$

so $\frac{du}{dt} = 64$ feet per second (at time 10).

(e) A small funnel in the shape of a cone is being emptied of fluid at the rate of 12 cubic centimeters per second (the tip of the cone is downward). The height of the cone is 20 cm and the radius of the top is 4 cm. How fast is the fluid level dropping when the level stands 5 cm above the vertex of the cone [The volume of a cone is $V = \frac{1}{3}\pi r^2 h$].

Draw a picture of an inverted cone. The radius at the top is 4, and the overall height is 20. Inside the cone, draw some water at a height of $h(t)$, with radius $r(t)$.

We are given information about the rate of change of volume of water, so we are given that $\frac{dV}{dt} = -12$. Note that the formula for volume is given in terms of r and h, but we only want $\frac{dh}{dt}$. We need a relationship between r and h...

You should see similar triangles (Draw a line right through the center of the cone. This, and the line forming the top radius are the two legs. The outer edge of the cone forms the hypotenuse).

$$\frac{\text{radius of top}}{\text{radius of water level}} = \frac{\text{overall height}}{\text{height of water}} \Rightarrow \frac{4}{r} = \frac{20}{h}$$

so that $r = \frac{h}{5}$. Substituting this into the formula for the volume will give the volume in terms of h alone:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{h}{5})^2 h = \frac{1}{75}\pi h^3$$

Now,

$$\frac{dV}{dt} = \frac{3\pi}{75}h^2\frac{dh}{dt}$$

and we know $\frac{dV}{dt} = -12$, $h = 5$, so

$$\frac{dh}{dt} = \frac{-12}{\pi}$$

(f) A balloon is being inflated by a pump at the rate of 2 cubic inches per second. How fast is the diameter changing when the radius is $\frac{1}{2}$ inch?
The volume is $V = \frac{4}{3}\pi r^3$ (this formula would be given to you on an exam/quiz). If we let h be the diameter, then we know that $2r = h$, so we can make V depend on diameter instead of radius:

$$V = \frac{4}{3}\pi(\frac{h}{2})^3 = \frac{\pi}{6}h^3$$

Now, the question is asking for $\frac{dh}{dt}$ when $h = 1$, and we are given that $\frac{dV}{dt} = 2$. Differentiate, and

$$\frac{dV}{dt} = \frac{\pi}{6}3h^2\frac{dh}{dt} = \frac{\pi}{2}h^2\frac{dh}{dt}$$

so that $\frac{dh}{dt} = \frac{4}{\pi}$.

(g) A particle moves on the hyperbola $x^2 - 18y^2 = 9$ in such a way that its y coordinate increases at a constant rate of 9 units per second. How fast is the $x-$coordinate changing when $x = 9$?
In this example, we don't need any labels. The question is to find $\frac{dx}{dt}$ when $\frac{dy}{dt} = 9$. Differentiate to get:

$$2x\frac{dx}{dt} - 36y\frac{dy}{dt} = 0$$

We're going to need the y value when $x = 9$, so go back to the original equation:

$$9^2 - 18y^2 = 9 \Rightarrow y = \pm 2$$

so we need to consider 2 $y-$values. Putting these into our derivative, we get:

$$2 \cdot 9 \cdot \frac{dx}{dt} - 36 \cdot 2 \cdot 9 = 0$$

so that $\frac{dx}{dt} = 36$. Put in $y = -2$ to get the second value of $\frac{dx}{dt} = -36$

(h) An object moves along the graph of $y = f(x)$. At a certain point, the slope of the curve is $\frac{1}{2}$ and the $x-$coordinate is decreasing at 3 units per second. At that point, how fast is the $y-$coordinate changing?
The key point here is that we want to think about both x and y as functions of t, so that when we differentiate, we get (use chain rule):

$$\frac{dy}{dt} = f'(x)\frac{dx}{dt}$$

The slope of the curve at a certain point is $f'(x) = \frac{1}{2}$, and $\frac{dx}{dt} = -3$, so we can plug these values in to get the change in y (with respect to time):

$$\frac{dy}{dt} = \frac{1}{2} \cdot (-3) = \frac{-3}{2}$$

(i) A rectangular trough is 8 feet long, 2 feet across the top, and 4 feet deep. If water flows in at a rate of 2 cubic feet per minute, how fast is the surface rising when the water is 1 foot deep?
The trough is a rectangular box. Let $x(t)$ be the height of the water at time t. Then the volume of the water is:

$$V = 16x \Rightarrow \frac{dV}{dt} = 16\frac{dx}{dt}$$

Put in $\frac{dV}{dt} = 2$ to get that $\frac{dx}{dt} = \frac{1}{8}$.

(j) If a mothball (sphere) evaporates at a rate proportional to its surface area $4\pi r^2$, show that its radius decreases at a constant rate.

Let $V(t)$ be the volume at time t. We are told that

$$\frac{dV}{dt} = kA(t) = k4\pi r^2$$

We want to show that $\frac{dr}{dt}$ is constant.

By the formula for $V(t) = \frac{4}{3}\pi r^3$, we know that:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Now compare the two formulas for $\frac{dV}{dt}$, and we see that $\frac{dr}{dt} = k$, which was the constant of proportionality!

(k) If an object is moving along the curve $y = x^3$, at what point(s) is the $y-$coordinate changing 3 times more rapidly than the $x-$coordinate?

Let's differentiate:

$$\frac{dy}{dt} = 3x^2 \frac{dx}{dt}$$

From this, we see that if we want $\frac{dy}{dt} = 3\frac{dx}{dt}$, then we must have $x = \pm 1$. We also could have $x = 0, y = 0$, since 0 is 3 times 0. All the points on the curve are therefore:

$$(0,0), (-1,1), (1,1)$$