Calculus I Summary

THE LIMIT

1. Definitions:

$$\lim_{x\to a^+} f(x) = L, \qquad \lim_{x\to a^-} f(x) = L$$

$$\lim_{x\to a} f(x) = L$$

2. Algebraic Methods:
   (a) Factor and Cancel
   (b) Multiply by Conjugate
   (c) Divide by x^n (Mainly for $x \to \infty$)
   (d) L'Hospital's Rule: For $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$

   $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

   [CANNOT be used to compute the derivative using limits!]
   i. If form is $0 \cdot \infty$, form $\frac{f}{\frac{1}{g}}$ to get the $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.
   ii. If form is $f(x)^{g(x)}$, rewrite so that

   $$f(x)^{g(x)} = \mathrm{e}^{g(x)\ln(f(x))}$$

   then take the limit of $g(x)\ln(f(x))$.

3. Horizontal/Vertical Asymptotes:
   (a) $x = a$ is a vertical asymptote for $f(x)$ if one of the following limits is infinite: $\lim_{x\to a^\pm} f(x)$
   (b) $y = b$ is a horizontal asymptote for $f(x)$ if one of the following is true: $\lim_{x\to\pm\infty} f(x) = b$

4. Heuristics that can be used:
   (a) "$\infty + \infty = \infty$", but $\infty - \infty$ is not necessarily 0.
   (b) If the denominator goes to zero, but the numerator does not, the limit is $\pm\infty$.
   (c) If the denominator goes to $\pm\infty$, and the numerator does not, the overall limit goes to zero.

5. Limit Laws (Sect 2.3)

6. **Continuity**
   (a) Definition:

   $$\lim_{x\to a} f(x) = f(a)$$

   Means 3 things: (1) $f(a)$ exists, (2) The limit exists, and (3) Items 1 and 2 are the same number.
   (b) IVT: If f is continuous on $[a, b]$, and w is a number between $f(a)$ and $f(b)$, there is at least one c in $[a, b]$ so that $f(c) = w$.
   Interpretations:
   i. If f is continuous, the range of a closed interval is an interval.
   ii. If f is continuous, and $f(x_1) > 0, f(x_2) < 0$, then there is a c between x_1 and x_2 where $f(c) = 0$ (f has at least one root in the interval between x_1 and x_2).
   (c) Continuous v. Differentiable: If f is differentiable at $x = a$, it is continuous at $x = a$. If f is continuous at $x = a$, we don't know if it is differentiable at $x = a$. That is, "All differentiable functions are continuous, but not all continuous functions are differentiable".

THE DERIVATIVE

1. Definition:
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
2. Rules for Differentiation (See table)
3. Higher derivatives: (a) Differential Equations, (b) Velocity and Acceleration, (c) Inc/Dec, concave up/down
4. Derivative of Inverses:
   (a) Be able to derive the derivative formula for the inverse trig functions (we needed implicit differentiation and triangles for this).
   (b) If the point (a, b) is on the graph of f, then: (1) the point (b, a) is on the graph of f^{-1}, and (2) $f'(b) = \frac{1}{f'(a)}$.
5. Equation of the Tangent Line: Please remember that $f'(x)$ gives a *FORMULA* for the slope, and is not the slope itself!!
6. Implicit Differentiation: Idea is to think of y as some function of x, and differentiate. For example, differentiate with respect to x: $x^2 + y^2 = 1 \Rightarrow 2x + 2y \cdot \frac{dy}{dx} = 0$

USING THE DERIVATIVE

1. MVT: If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a c in the interval (a, b) so that
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
   Interpretations and Useage:
   (a) There is a tangent line with the same slope as the line between the points $(a, f(a))$ and $(b, f(b))$.
   (b) If a car has an average (mean) velocity of N mph, then at some point, the speedometer read exactly N.
   (c) If f is differentiable, and $f'(x) = 0$ has n solutions, there can be at most $n+1$ solutions to $f(x) = 0$ (Between any two roots of f, f' must be zero).
2. Differentials:
$$dy = f'(x)\, dx$$
   This is thinking of the derivative as a *multiplier*- For example, if $f(x) = x^3$, then $dy = 3x^2\, dx$. So if $x = 1$, then the change in y is approximately 3 times the change in x. At $x = 2$, the change in y is approximately 12 times the change in x, etc.
3. Linear Approximations:
   (a) Given f, x, and Δx, then
$$\Delta y = f(x + \Delta x) - f(x)$$
   (b) Use dy to approximate Δy. We always assume $dx = \Delta x$, since this is the independent variable.
   (c) The **linearization** of f at $x = a$ is the tangent line approximation to f at $x = a$: $y - f(a) = f'(a)(x - a)$.
   Connection: If we think of x as $a + \Delta a$, then we could write the equation of the tangent line as
$$\Delta y = f'(a)\Delta a$$
   and note the similarity to: $dy = f'(x)\, dx$

4. Max's and Min's
   (a) EVT: If f is continuous on $[a, b]$, then f will attain a global max and global min on $[a, b]$. These points will be either at critical points or at endpoints.
   (b) Local: Let $x = a$ be a critical point of f.
      i. First Derivative Test:
         A. If $f'(x)$ changes sign at $x = a$, then we have a local min (if changes from $-$ to $+$), or a local max (if changes from $+$ to $-$).
         B. If $f'(x)$ does not change sign at $x = a$, we have neither.
      ii. Second Derivative Test: Let $f'(a) = 0$. Then
         A. If $f''(a) > 0$, we have a local min at $x = a$.
         B. If $f''(a) < 0$, we have a local max at $x = a$.
         C. If $f''(a) = 0$, the test is inconclusive.
   (c) Global:
      i. On a closed interval: Use the EVT, build a chart using endpoints and critical points.
      ii. Not a closed interval: If $f'(x)$ only changes sign once at $x = a$, then $x = a, y = f(a)$ is either a global min ($-$ to $+$) or a global max ($+$ to $-$).
5. Related Rates: Main idea here was to think of all variables as depending on time. Relate the derivative of one quantity to the derivative of the other quantity. Similar triangles and the Pythagorean Theorem played a big role here.
6. Newton's Method: Used to solve $f(x) = 0$ for x. You begin with an initial guess: x_0, and refine this estimate:
$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$
   Graphically, these values are the $x-$intercepts of the tangent lines to f at $(x_i, f(x_i))$.
7. Antidifferentiation: See the Table.

Miscellaneous Issues

(basically pre-calculus activities)

1. You should know the quadratic formula, formulas for the area and circumference of a circle, a rectangle, a square and a triangle.
2. You should be able to compute the 6 trig functions of the special angles: $\frac{\pi}{4}$, $\frac{\pi}{6}$, $\frac{\pi}{3}$. You should be able to use the unit circle to compute the sine and cosine of $0, \pi, 2\pi$ and translate the special angles to different quadrants.
3. Using the unit circle, be able to compute the inverse sine, cosine and tangent using the special angles.
4. Given the graph of $y = f(x)$, give the graph of $f^{-1}(x)$, $f'(x)$, and $F(x)$ (where F is an antiderivative of f).
5. Remember that the notation $f^{-1}(x)$ is reserved ONLY for inverses. This is not the same as $\frac{1}{f(x)}$.
6. Be able to find a formula for $f^{-1}(x)$ given the formula for $f(x)$.
7. Rules of logarithms and exponents
8. Be able to re-write the absolute value function as a piece-wise defined function. We needed to use sign charts here.

Differentiation Table

1. Notation: These are all equivalent

$$y',\ f'(x),\ \frac{dy}{dx},\ \frac{d}{dx}\text{(expression in } x\text{)}\ , Df$$

2. Tables:

f	f'
cf	cf'
$f \pm g$	$f' \pm g'$
fg	$f'g + fg'$
$f(g(x))$	$f'(g(x))g'(x)$
$\frac{f}{g}$	$\frac{f'g - fg'}{g^2}$
$f(x)^{g(x)}$	Use log diff.

f	f'
c	0
x^n	nx^{n-1}
e^x	e^x
a^x	$a^x \ln(a)$
$\ln\lvert x\rvert$	$\frac{1}{x}$
$\log_a(x)$	$\frac{1}{x} \cdot \frac{1}{\ln(a)}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\cot(x)$	$-\csc^2(x)$
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1}(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$

Antidifferentiation Notes

(to be continued in Calculus II)

1. Definition: $F(x)$ is an antiderivative of $f(x)$ if $F'(x) = f(x)$
2. If $F(x)$ is an antiderivative of $f(x)$, so is $F(x) + C$, for any constant C.
3. You can always check your antiderivative by differentiating it.
4. Tables:

f	F
cf	cF
$f \pm g$	$F \pm G$

f	F
x^n	$\frac{1}{n+1}x^{n+1}$, for $n \neq -1$
$\frac{1}{x}$	$\ln\lvert x\rvert$
$\sin(x)$	$-\cos(x)$
$\cos(x)$	$\sin(x)$
$\sec^2(x)$	$\tan(x)$
$\sec(x)\tan(x)$	$\sec(x)$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}(x)$
$\frac{1}{1+x^2}$	$\tan^{-1}(x)$
e^x	e^x
a^x	$\frac{1}{\ln(a)}a^x$