Optimization Problems

Solutions

1. Birds are flying from the island to the nest as noted in the figure. The island is 8 km from the straight shoreline, (to point B), and the nest is 11 km from point B. It takes 8/5 as much energy to fly over the water as to fly over the land. At what point on the shore should the birds fly to minimize the overall amount of energy?

   SOLUTION: The following solution labeled the distance from the nest to the point on the shore as y. Therefore, the distance from the island to the point on the shore is given as $\sqrt{(11-y)^2+64}$. Given this, the energy:

   $$E(y) = \frac{8}{5}\sqrt{(11-y)^2+64} + y \qquad 0 \le y \le 11$$

   Differentiating:

   $$\frac{dE}{dy} = \frac{8}{5} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{(11-y)^2+64}} \cdot 2(11-y)(-1) + 1$$

   Solving for the critical points,

   $$\frac{-8(11-y)}{5\sqrt{(11-y)^2+64}} = -1 \quad \Rightarrow \quad 8(11-y) = 5\sqrt{(11-y)^2+64} \quad \Rightarrow$$

   $$64(11-y)^2 = 25((11-y)^2+64) \quad \Rightarrow \quad 39(11-y)^2 = 25 \cdot 64$$

   Solve this to get:

   $$y = 11 - \frac{40}{\sqrt{39}} \approx 4.59487$$

   Put these values into E to find the minimum:

y	0	4.59487	11
$E(y)$	21.76	23.8	20.991

   So we should land at the point we found, approximately 4.59487, measured from the nest, or 6.4051 measured from a straight line to the island.

   (A brief alternative) If your function was:

   $$E(x) = \frac{8}{5}\sqrt{x^2+64} + (11-x), \qquad 0 \le x \le 11$$

   then

   $$\frac{dE}{dx} = \frac{8}{5} \cdot \frac{x}{x^2+64} - 1$$

   and the critical point is $40/\sqrt{39} \approx 6.40512$. This leads to the same conclusion.

2. A conical drinking cup is to be made from a circular piece of paper with a sector cut as shown. The radius of the circle is 5 cm. Find the maximum capacity of the cup.

From class, we saw that we could write the volume as:

$$V(r) = \frac{1}{3}\pi r^2 \sqrt{25 - r^2}, \qquad 0 \le r \le 5$$

Finding the critical points,

$$\frac{dV}{dr} = \frac{2}{3}\pi r\sqrt{25 - r^2} - \frac{\pi}{3} \cdot \frac{r^3}{\sqrt{25 - r^2}} = 0$$

Solve for r:

$$\frac{2\pi}{3} r\sqrt{25 - r^2} = \frac{\pi}{3} \cdot \frac{r^3}{\sqrt{25 - r^2}}$$

Divide both sides by π, multiply by 3 and multiply by $\sqrt{25 - r^2}$:

$$2r(25 - r^2) = r^3 \qquad \Rightarrow \qquad 50r - 2r^3 = r^3 \quad \Rightarrow \quad 50r = 3r^3$$

Either $r = 0$ or $50 = 3r^2$, in which case $r = \sqrt{50/3}$ (ignore the negative root, since $r \ge 0$).

Now we can build our table:

r	V
0	0
$\sqrt{50/3}$	50.3833
5	0

So the maximum capacity of the cup is approximately 50.3833.

Something Extra: How exactly would you cut out the sector in order to get this optimal radius? The circumference of the top of the cup is the outer (partial) circumference of the circular piece of paper. If x is the length of the sector cut out, then:

$$50\pi - x$$

is the circumference of the top of the cup. That gives:

$$50\pi - x = 2\pi r$$

so the length of the sector to cut out:

$$x = 50\pi - 2\pi r = 50\pi - 2\pi\sqrt{50/3} \approx 131.4$$

3. A painting in an art gallery has a hieght of 57 cm. The painting is hung 14 cm above the eye level of a person viewing the painting. How far should the observer stand from the painting to get the best view? (The best view is the maximum value of θ, shown in the figure).

First, we had to introduce a new variable, α, so that the full angle, $\theta + \alpha$ was inside a right triangle.

From the picture, we get:

$$\tan(\theta + \alpha) = \frac{57 + 14}{x} = \frac{71}{x}$$

and

$$\tan(\alpha) = \frac{14}{x}$$

We would like to write θ as a function of x. From the first equation,

$$\theta + \alpha = \tan^{-1}\left(\frac{71}{x}\right) \quad \Rightarrow \quad \theta = \tan^{-1}\left(\frac{71}{x}\right) - \alpha$$

Substitute the value for $\alpha = \tan^{-1}(14/x)$:

$$\theta = \tan^{-1}\left(\frac{71}{x}\right) - \tan^{-1}\left(\frac{14}{x}\right)$$

we also note that $x > 0$ (so we are maximizing an unbounded function).

Differentiate to solve for the critical points of θ:

$$\frac{d\theta}{dx} = \frac{1}{1 + (71/x)^2} \cdot \frac{-71}{x^2} - \frac{1}{1 + (14/x)^2} \cdot \frac{-14}{x^2}$$

Now simplify and set equal to zero:

$$\frac{d\theta}{dt} = \frac{1}{\frac{x^2+5041}{x^2}} \cdot \frac{-71}{x^2} - \frac{1}{\frac{x^2+196}{x^2}} \cdot \frac{-14}{x^2}$$

$$\frac{d\theta}{dt} = \frac{-71}{x^2 + 5041} + \frac{14}{x^2 + 196} = 0$$

$$\frac{71}{x^2 + 5041} = \frac{14}{x^2 + 196} \quad \Rightarrow \quad 71(x^2 + 196) = 14(x^2 + 5041)$$

so that

$$57x^2 = 56658 \quad \Rightarrow \quad x^2 = 994 \quad \Rightarrow \quad x = \sqrt{994} \approx 31.53$$

Check to see how the sign of $d\theta/dt$ changes:

$d\theta/dt$	$+$	$-$
	$0 < x < 31.53$	$x > 31.53$

Therefore, we have a maximizer.

4. The illumination of an object by a light source is directly proportional to the strength of the source, and inversely proportional to the square of the distance to the source.

   Consider two light sources, where one source has three times the strength of the other. On the line between them, where should an object be placed to minimize its illumination? The sources are 20 units apart.

   If k is the strength of the weaker light source, the overall illumination on the object is:

$$Illum(x) = \frac{3k}{x} + \frac{k}{20-x}$$

   where x is the distance to the stronger source (Note that there are many ways to set this up, this is one example).

   We also note that $0 < x < 20$, but from the function, we see that, as $x \to 0^+$ or $x \to 20^-$, the value of $Illum(x) \to \infty$. Therefore, there is a global minimum on this interval.

   To find it, we look for the critical points:

$$Illum'(x) = \frac{-3k}{x^2} + \frac{k}{(20-x)^2} = 0$$

   so that

$$\frac{1}{(20-x)^2} = \frac{3}{x^2} \quad \Rightarrow \quad x^2 = 3(20-x)^2 \quad \Rightarrow \quad x^2 = 3(400 - 40x + x^2)$$

   or

$$0 = 1200 - 120x + 2x^2 \qquad x^2 - 60x + 600 = 0$$

   From the quadratic formula,

$$x = 30 - 10\sqrt{3} \approx 12.69$$

   So the position of the object should be 12.69 units away from the stronger source, or 7.31 units from the weaker source.

5. There is a 6 foot fence that is 8 feet from a building. What is the length of the shortest ladder that can be used to reach over the fence onto the building?

   If we let h be the height from the ground to the top of the ladder against the building, and x be the length from the bottom of the ladder to the fence, then the length squared of the ladder is:

$$F = L^2 = h^2 + (8+x)^2$$

   It suffices to find the minimum of this quantity (as we'll show). To get h in terms of x, use similar triangles:

$$\frac{h}{8+x} = \frac{6}{x} \quad \Rightarrow \quad h = \frac{6(8+x)}{x}$$

   Now,

$$F = \frac{36(8+x)^2}{x^2} + (8+x)^2$$

   where $x > 0$. Differentiate:

$$\frac{dF}{dx} = \frac{72(8+x)x^2 - 36(8+x)^2 \cdot 2x}{x^4} + 2(8+x)$$

   Simplify by factoring:

$$\frac{dF}{dx} = 2(8+x)\left[\frac{36x^2 - 36(8+x)x}{x^4} + 1\right] = 2(8+x)\left[\frac{36(-8x)}{x^4} + 1\right]$$

   so that

$$\frac{dF}{dx} = 2(8+x) \cdot \frac{-288x + x^4}{x^4} = 2(8+x)\frac{x^3 - 288}{x^3}$$

   Therefore, the critical points are $x = -8$, $x = \sqrt[3]{288} \approx 6.6038$, or $x = 0$. Since $x > 0$, the only critical point is $x \approx 6.6038$.

   We need to be sure that this is a minimum:

$\frac{dF}{dx}$	$-$	$+$
	$0 < x < 6.6038$	$x > 6.6038$

6. You launch your boat from point A on one bank of a straight river that is 3 km wide. You want to reach point B as quickly as possible.

   You can row at 6 kph, and run at 8 kph. At what point along the bank should you land in order to minimize the time it takes you to go from point A to point B?

   We'll be using the rule that: distance = rate × time

   Some initial observations:

   - If you row all the way, the distance is $\sqrt{8^2+3^2} \approx 8.544$, so that the time is $8.544/6 \approx 1.424$ hours
   - If you go straight across the river, and run the rest of the way, you will row 3 km, run 8km, and the total time will be:

$$\frac{3}{6} + \frac{8}{8} = 1.5 \text{ hours}$$

   If we row x km and run y km, then the total time to minimize will be:

$$\frac{x}{6} + \frac{y}{8}$$

   If we write this as a function of x (the distance across the river), then $y = 8 - \sqrt{x^2-9}$. Therefore,

$$F(x) = \frac{x}{6} + \frac{8-\sqrt{x^2-9}}{8} = \frac{1}{6}x + 1 - \frac{1}{8}\sqrt{x^2-9}$$

   Now differentiate and find x so that F is a minimum. We note that $3 \le x \le \sqrt{8^2+3^2}$ (these are the extreme cases already considered).

   Now,

$$F'(x) = \frac{1}{6} - \frac{1}{8}\cdot\frac{1}{2}(x^2-9)^{-1/2}(2x) = \frac{1}{6} - \frac{x}{8\sqrt{x^2-9}}$$

   Now set this to zero and solve for x:

$$\frac{1}{6} = \frac{x}{8\sqrt{x^2-9}} \quad \Rightarrow \quad 8\sqrt{x^2-9} = 6x \quad \Rightarrow \quad 64(x^2-9) = 36x^2$$

$$28x^2 = 576 \quad \Rightarrow \quad x^2 = \frac{144}{7} \quad \Rightarrow \quad x = \frac{12}{\sqrt{7}} \approx 4.536$$

   Put this into a table:

x	$F(x)$
3	1.5
4.536	1.3307
8.544	1.4240

   Conclusion: To get the minimum time, we should launch the boat so as to land y units from point B, which is approximately 4.5983 km north of B. Alternatively, we could say that we need to launch the boat so as to land after 4.536 km on the water (that's x).