Exam 1 Review Solutions

Please also review the old quizzes, and be sure that you understand the homework problems. General notes: (1) Always give an algebraic reason for your answer (graphs are not sufficient), (2) You may not use shortcut rules for derivatives (these are covered in Chapter 3- If you don't know what I'm talking about, don't worry about it). (3) We will usually use the a, h version of the definition of the derivative- It's usually easier with the algebra.

1. Finish the definition:

   (a) $\lim_{x\to a} f(x) = L$ if, for every $\epsilon > 0$, there is a $\delta > 0$ such that: if $0 < |x-a| < \delta$, then $|f(x) - L| < \epsilon$

   (b) The function f is continuous at $x = a$ if: $\lim_{x\to a} f(x) = f(a)$.

   Also, this definition implies three things: (i) $f(a)$ exists, (ii) $\lim_{x\to a} f(x)$ exists, (iii) Items (i) and (ii) give the save value.

   (c) The derivative of f at the point $x = a$ is:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

2. Find the domain: $f(x) = \sqrt{\dfrac{x^2-4}{1-x^2}}$

   To find the domain, we require that:

$$\frac{(x+2)(x-2)}{(1+x)(1-x)} \geq 0$$

   so we use a table to solve:

$(x+2)$	$-$	$+$	$+$	$+$	$+$
$(x-2)$	$-$	$-$	$-$	$-$	$+$
$(1+x)$	$-$	$-$	$+$	$+$	$+$
$(1-x)$	$+$	$+$	$+$	$-$	$-$
	$x < -2$	$-2 < x < -1$	$-1 < x < 1$	$1 < x < 2$	$x > 2$

   By the table, the domain is:

   $-2 \leq x < -1$, or $1 < x \leq 2$, (which is also where $f(x)$ is continuous). (NOTE: It is easier to type a sign chart using the intervals under each column- When you're writing by hand, that is not necessary- Just label the where the breaks are on the number line)

3. Find the domain: $f(x) = \ln(x^2 + 2x - 3)$

   The domain of the natural log is the set of positive real numbers (not including zero). So we need to solve for x so that

$$x^2 + 2x - 3 > 0 \quad \Rightarrow \quad (x+3)(x-1) > 0$$

   Use a sign chart:

$(x+3)$	$-$	$+$	$+$
$(x-1)$	$-$	$-$	$+$
	$x < -3$	$-3 < x < 1$	$x > 1$

   So $x < -3$ or $x > 1$.

4. The formula for the equation of the tangent line to $f(x)$ at $x = a$ is (use the point-slope form):

   The point that the line must go through is $(a, f(a))$. The slope of the tangent line is denoted by $f'(a)$. Put these together (with the point-slope form for a line) to get:

$$y - f(a) = f'(a)(x - a)$$

5. True or False, and give a short reason:

(a) $\lim_{x\to 4}\left(\frac{2x}{x-4}-\frac{8}{x-4}\right)=\lim_{x\to 4}\left(\frac{2x}{x-4}\right)-\lim_{x\to 4}\left(\frac{8}{x-4}\right)$

FALSE. The limit rules (Section 2.3) say that we can only distribute the limit through a sum *if the two limits exist separately.* In this case, the two limits do NOT exist separately.

(b) $\lim_{x\to 1}\frac{x+4}{x^2-3}=\frac{\lim_{x\to 1}(x+4)}{\lim_{x\to 1}(x^2-3)}$

TRUE. The limit rules say we can do this as long as the limit in the numerator and denominator exist (and the limit in the denominator is not zero). By this rule, the limit is $(5/-2)=-2.5$.

By the way, this is how we showed that all rational functions are continuous on its domain.

(c) If f is continuous and $f(1)=3$, $f(2)=4$, then there is an r so that $f(r)=\pi$.

TRUE. Since $3\le\pi\le 4$ (or $f(1)\le\pi\le f(2)$), and the function f is continuous, the Intermediate Value Theorem says that there is a c in the interval $[1,2]$ so that $f(c)=\pi$.

(d) All functions are continuous on their respective domains.

FALSE. It is easy to construct a function that is not continuous on its domain- We can use a piecewise defined function, like:

$$f(x)=\begin{cases} x+1 & \text{if } x>1\\ 0 & \text{if } x\le 1\end{cases}$$

and so, while $f(1)=0$, the limit at $x=1$ does not exist.

(e) $\ln(x+3)=\ln(x)+\ln(3)$

FALSE. We do have formulas for $\ln(ab)=\ln(a)+\ln(b)$, but we do NOT have formulas for $\ln(a+b)$.

(f) $\sin^{-1}(x)=\frac{1}{\sin(x)}$

FALSE. We reserve the -1 exponent in functions to denote the inverse of the function, not the reciprocal. For example, here the inverse sine function has domain $-1\le x\le 1$, but the domain of $1/sin(x)=csc(x)$ are angles (excluding the points where $sin(x)=0$)- so the domains do not match as well.

(g) A vertical line intersects the graph of a function at most once.

TRUE. This is the *vertical line test* to see if a graph is the graph of a function. That is, each x can must have a unique $f(x)$.

(h) If, when taking a limit of a rational function, we get $\frac{0}{0}$, then the limit does not exist.

FALSE. If we get $0/0$, we cannot say if the limit exists or not (we must do more work).

6. For the function $f(x)=x^2$, find and simplify the expression

$$\frac{f(x+h)-f(x-2h)}{3h}$$

If $f(x)=x^2$, then $f(x+h)=(x+h)^2=x^2+2xh+h^2$, and $f(x-2h)=(x-2h)^2=x^2-4xh+4h^2$. Putting these into the given expression,

$$\frac{x^2+2xh+h^2-(x^2-4xh+4h^2)}{3h}=\frac{6xh-3h^2}{3h}=2x-h$$

7. Show that there must be at least one real solution to:

$$x^5=x^2+4$$

We need to set this up for the Intermediate Value Theorem, so we need to re-write the equation so that $f(x)=0$. In this case, we get:

$$x^5-x^2-4=0$$

so that $f(x)=x^5-x^2-4$, which is a polynomial, and therefore continuous for all x (continuity was a requirement of the IVT).

Now we begin substituting values of x into f until we observe a sign change:

$$f(0) = -4 \quad f(1) = -4 \quad f(2) = 24$$

so the IVT says there is at least one c in $[1,2]$ so that $f(c) = 0$. This will be the solution to the original equation.

8. Solve for x:

   (a) $x^2 < 2x + 8$
   Rewrite, factor, and use a sign chart: $x^2 - 2x - 8 < 0 \Rightarrow (x-4)(x+2) < 0$

$x-4$	$-$	$-$	$+$
$x+2$	$-$	$+$	$+$
	$x < -2$	$-2 < x < 4$	$x > 4$

   The solution is the set of x so that $-2 < x < 4$.

   (b) $e^{x^2} = 4$
   Take the natural log of both sides (or rewrite using logs):

$$\ln(4) = x^2 \quad x = \pm\sqrt{\ln(4)}$$

   (We do note that $\ln(4) > 0$, so this is a real solution).

   (c) $\ln(5 - 2x) = -3$
   Rewrite in exponential form:

$$e^{-3} = 5 - 2x \quad \Rightarrow \quad x = \frac{5 - e^{-3}}{2}$$

   (d) $\ln(\ln(x)) = 1$
   Rewrite in exponential form twice:

$$e^1 = \ln(x) \Rightarrow e^e = x$$

   We might verify that this is a solution:

$$\ln(\ln(e^e)) = \ln(e\ln(e)) = \ln(e \cdot 1) = \ln(e) = 1$$

9. If $f(x) = 1 - x^3$, $g(x) = \frac{1}{x}$, compute the expression for $f \circ g$, $g \circ g$ (and simplify), $g \circ f$.

$$f \circ g = f(g(x)) = f(1/x) = 1 - (1/x)^3 = 1 - \frac{1}{x^3}$$

$$g \circ g = \frac{1}{\frac{1}{x}} = x$$

$$g \circ f = g(1 - x^3) = \frac{1}{1 - x^3}$$

10. Use the ϵ, δ definition of the limit to show that $\lim_{x \to 1} \left(3 - \frac{5x}{2}\right) = \frac{1}{2}$

   We begin with $|f(x) - L| < \epsilon$, and we want to end with $|x - a| <$"Expression in ϵ", which will be δ. That is, we want to find a formula for δ given any ϵ. Here we go:

$$\left|3 - \frac{5x}{2} - \frac{1}{2}\right| < \epsilon \quad \Rightarrow \left|\frac{5}{2} - \frac{5x}{2}\right| < \epsilon \quad \Rightarrow \frac{5}{2}|1 - x| < \epsilon \quad \Rightarrow |1 - x| < \frac{2\epsilon}{5}$$

   Is it true that $|1 - x| = |-(1 - x)| = |x - 1|$? Yes- the absolute value of a number is the same as the absolute value of the negative number.

   CONCLUSION: Given any $\epsilon > 0$, if $\delta = \frac{2\epsilon}{5}$, then $0 < |x - 1| < \delta$ will guarantee that $|f(x) - (1/2)| < \epsilon$.

11. Compute each limit algebraically (if it exists):

(a) $\lim_{h\to 0} \dfrac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$

$$\lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x}\sqrt{x+h}} \cdot \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} =$$

$$\lim_{h\to 0} \frac{1}{h} \cdot \frac{x - (x+h)}{\sqrt{x}\sqrt{x+h}(\sqrt{x} + \sqrt{x+h})} = \frac{-1}{x \cdot 2\sqrt{x}} = \frac{-1}{2x^{3/2}}$$

(b) $\lim_{x\to 2} \dfrac{x^2-4}{\frac{1}{x} - \frac{1}{2}}$

$$\lim_{x\to 2} \frac{x^2-4}{\frac{1}{x} - \frac{1}{2}} = \lim_{x\to 2} \frac{x^2-4}{\frac{2-x}{2x}} = \lim_{x\to 2} \frac{(x+2)(x-2)2x}{2-x} = \lim_{x\to 2} -(x+2)2x = -16$$

(c) $\lim_{x\to 2} \dfrac{x^2-2x-3}{x-3}$

$$\lim_{x\to 2} \frac{x^2-2x-3}{x-3} = \frac{4-4-3}{2-3} = 3$$

(Be sure to check the numbers before doing any algebra!)

(d) $\lim_{x\to\infty} (\sqrt{x^2+2x} - x)$

$$\lim_{x\to\infty} \sqrt{x^2+2x} - x \cdot \frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x} = \lim_{x\to\infty} \frac{(x^2+2x)-x^2}{\sqrt{x^2+2x}+x} = \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+2x}+x} =$$

$$\lim_{x\to\infty} \frac{2x/x}{(\sqrt{x^2+2x}+x)/x} = \lim_{x\to\infty} \frac{2}{\sqrt{\frac{x^2+2x}{x^2}}+1} = \lim_{x\to\infty} \frac{2}{\sqrt{1+\frac{2}{x}}+1} = \frac{2}{2} = 1$$

Remember, if $x > 0$, we can write $x = \sqrt{x^2}$, and if $x < 0$ (in the next problem), we will need to write $x = -\sqrt{x^2}$.

(e) $\lim_{x\to -\infty} \dfrac{3x+2}{\sqrt{x^2-1}}$

$$\lim_{x\to -\infty} \frac{3x+2}{\sqrt{x^2-1}} = \lim_{x\to -\infty} \frac{3+\frac{2}{x}}{(\sqrt{x^2-1})/x} = \lim_{x\to -\infty} \frac{3+\frac{2}{x}}{-\sqrt{\frac{x^2-1}{x^2}}} = \lim_{x\to -\infty} \frac{3+\frac{2}{x}}{-\sqrt{1-\frac{1}{x^2}}} = -3$$

(f) $\lim_{x\to 0} x^2 \sin\left(\dfrac{x^2-5x}{x^{26}}\right)$

We can think of this as: $\lim_{x\to 0} x^2 \sin(A)$ I don't care what A is, as long as it is going to infinity and not a number. If A were going to a number as $x \to 0$, then the limit could be found by substitution of $x = 0$. Since A is going to infinity, we can use the Squeeze Theorem:

$$-x^2 \le x^2 \sin(A) \le x^2$$

Because the left and right sides of the inequality go to zero, so must the middle function. (The answer is zero, by the squeeze theorem).

(g) $\lim_{x\to 2} \dfrac{x^2-4}{2x^2+3x-14}$

$$\lim_{x\to 2} \frac{x^2-4}{2x^2+3x-14} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(2x+7)(x-2)} = \frac{4}{11}$$

If you had trouble factoring the denominator, be sure to use the fact that we know that $(x-2)$ is a factor- That is, if $p(x)$ is a polynomial, and $p(a) = 0$, then $(x-a)$ is a factor for $p(x)$.

(h) $\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right)$
Before continuing, rewrite the function to get rid of the absolute values:

$$\frac{1}{x}-\frac{1}{|x|}=\begin{cases}0 & \text{if } x>0\\ \frac{2}{x} & \text{if } x<0\end{cases}$$

so the limit as $x\to 0^+$ is 0 .

12. Find all vertical and horizontal asymptotes for $\frac{2x+3}{\sqrt{x^2-2x-3}}$
Vertical asymptotes for a fraction like this will appear when the denominator is zero AND the numerator is NOT zero. Factoring the denominator, we get $x^2-2x-3=(x-3)(x+1)$. The numerator is not zero at either $x=3$ or $x=-1$, so this function will have two vertical asymptotes,

$$x=3,\quad x=-1$$

(RECALL that $x=a$ is a vertical asymptote for f if $\lim_{x\to a^\pm} f(x)=\pm\infty$)
For the horizontal asymptotes, we take the limit as $x\to\pm\infty$:

$$\lim_{x\to\infty}\frac{2x+3}{\sqrt{x^2-2x-3}}=\lim_{x\to\infty}\frac{2+\frac{3}{x}}{\sqrt{\frac{x^2-2x-3}{x^2}}}=\lim_{x\to\infty}\frac{2+\frac{3}{x}}{\sqrt{1-\frac{2}{x}-\frac{3}{x^2}}}=2$$

$$\lim_{x\to-\infty}\frac{2x+3}{\sqrt{x^2-2x-3}}=\lim_{x\to-\infty}\frac{2+\frac{3}{x}}{-\sqrt{\frac{x^2-2x-3}{x^2}}}=\lim_{x\to-\infty}\frac{2+\frac{3}{x}}{-\sqrt{1-\frac{2}{x}-\frac{3}{x^2}}}=-2$$

The lines $y=2$ and $y=-2$ are horizontal asymptotes.

13. Find all the values of a for which f will be continuous for all real values.

(a) $f(x)=\begin{cases}4-x^2 & \text{if } x\le 3\\ 3-ax & \text{if } x>3\end{cases}$
Go through all three parts for continuity! The only "problem" value(s) of x is at $x=3$ At this point:

- $f(3)=4-9=-5$, so $f(3)$ exists.
- The limit as $x\to 3$ has to be found separately from the right and left:

$$\lim_{x\to 3^+} 3-ax=3-3a$$

  and

$$\lim_{x\to 3^-} 4-x^2=-5$$

  Therefore, for the limit to exist, $3-3a=-5$, or $a=8/3$. At this value of a, the limit is -5.
- We see that, if $a=8/3$, then the first two numbers ($f(3)$ and the limit) are equal.

If $a=8/3$, the function is continuous for every x.

(b) $f(x)=\begin{cases}x^2-2 & \text{if } x\le a\\ 2x-1 & \text{if } x>a\end{cases}$
Again we see that, if $x<a$, $f(x)=x^2-2$, which is continuous. Also, if $x>a$, then $f(x)=2x-1$ is continuous. The only problem point will be at $x=a$. At this point, we check the three conditions for continuity:

- $f(a)=a^2-2$, so this exists for all a.
- The limit must be found separately from the right and left:

$$\lim_{x\to a^+} 2x-1=2a-1$$

$$\lim_{x\to a^-} x^2-2=a^2-2$$

For the limit to exist overall, we must have:

$$a^2 - 2 = 2a - 1 \quad \Rightarrow a^2 - 2a - 1 = 0 \quad \Rightarrow a = 1 \pm \sqrt{2}$$

The limit will exist if $a = 1 \pm \sqrt{2}$. In these cases, the limits will be $1 \pm 2\sqrt{2}$.

- If $a = 1 \pm \sqrt{2}$, then $f(a) = f(1 \pm \sqrt{2}) = 1 \pm 2\sqrt{2}$, which makes the first two items equal.

If a is either $1 + \sqrt{2}$ or $1 - \sqrt{2}$, we have shown that f is continuous at every x.

(c) $f(x) = \begin{cases} \frac{x^2-16}{x^2-4} & \text{if } x \neq \pm 2 \\ a & \text{if } x = \pm 2 \end{cases}$

There is no value of a that will make this function continuous, since the limit does not exist at $x = 2$ (that is, $(x-2)$ and $(x+2)$ will not cancel out of the fraction. For example,

$$\lim_{x \to 2^{\pm}} \frac{x^2 - 16}{x^2 - 4} = \pm\infty$$

(The denominator is going to zero but the numerator is NOT).

14. The displacement (signed distance) of an object moving in a straight line is given by $s(t) = 1 + 2t + t^2/4$, where t is in seconds.

(a) Find the average velocity over the time period $[1, 2]$.

The average velocity is $\dfrac{s(2) - s(1)}{2 - 1} = \dfrac{6 - \frac{13}{4}}{2 - 1} = \dfrac{11}{4}$

(b) Find the instantaneous velocity at $t = 1$.

The instantaneous velocity is $s'(1)$:

$$\lim_{h \to 0} \frac{s(1+h) - s(1)}{h} = \lim_{h \to 0} \frac{(1 + 2(1+h) + (1+h)^2/4) - (13/4)}{h} =$$

$$\lim_{h \to 0} \frac{\left(\frac{13}{4} + \frac{5}{2}h + \frac{h^2}{4}\right) - \frac{13}{4}}{h} = \lim_{h \to 0} \frac{h\left(\frac{5}{2} + \frac{h}{4}\right)}{h} = \frac{5}{2}$$

15. Find the equation of the tangent line to $y = \frac{2}{1-3x}$ at $x = 0$.

We need a point and a slope:

- The tangent line touches the function at $x = 0$. Put this into the formula for y to find the $y-$coordinates: $y = 2$. The line therefore goes through the point $(0, 2)$.
- The slope of the tangent line is the derivative at 0. In this case,

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{2}{1-3h} - 2}{h} = \lim_{h \to 0} \frac{\frac{2}{1-3h} - \frac{2(1-3h)}{1-3h}}{h} =$$

$$\lim_{h \to 0} \frac{2 - 2(1-3h)}{1-3h} \cdot \frac{1}{h} = \lim_{h \to 0} \frac{6h}{1-3h} \cdot \frac{1}{h} = \lim_{h \to 0} \frac{6}{1-3h} = 6$$

The line through $(0, 2)$ with slope 6 is the tangent line:

$$y - 2 = 6(x - 0) \text{ or } y = 6x + 2$$

16. Use the ϵ, δ definition of the limit to show that: $\lim\limits_{x \to 5} (7x - 27) = 8$

We need a formula for δ given any $\epsilon > 0$. To get it, start with $|f(x) - L| < \epsilon$ and simplify to $|x - a|$- In this case,

$$|7x - 27 - 8| < \epsilon \qquad \text{We want} \qquad |x - 5|$$

$$|7x - 35| < \epsilon \quad \Rightarrow \quad |x - 5| < \frac{\epsilon}{7}$$

Therefore, for any $\epsilon > 0$, there is a $\delta = \frac{\epsilon}{7}$ so that, if $|x - 5| < \delta$, then $|f(x) - 8| < \epsilon$.

17. Consider the function:

$$f(x) = \begin{cases} x+3, & \text{if } x < 1 \\ x^2 & \text{if } x \geq 1 \end{cases}$$

The limit of this function at $x = 1$ does not exist (why?). Draw a sketch of $f(x)$ and show how the ϵ, δ definition of the limit fails if we think the limit should be $L = 1$.

We'll look at the solution in class... The problem is that any number to the left of $x = 1$ will have corresponding $y-$ values that will approach $y = 1 + 3 = 4$ instead of $y = 1$ on the parabola. Therefore, if the ϵ is less than, say 1, we cannot find a δ so that all the corresponding y's are within ϵ of 1.

18. For each function below, compute the derivative $f'(a)$ (use the definition of the derivative).

(a) $f(x) = \sqrt{1+2x}$

$$\lim_{h\to 0} \frac{\sqrt{1+2(a+h)} - \sqrt{1+2a}}{h} = \lim_{h\to 0} \frac{\sqrt{1+2(a+h)} - \sqrt{1+2a}}{h} \cdot \frac{\sqrt{1+2(a+h)} + \sqrt{1+2a}}{\sqrt{1+2(a+h)} + \sqrt{1+2a}} =$$

$$\lim_{h\to 0} \frac{1+2a+2h-1-2a}{h(\sqrt{1+2(a+h)} + \sqrt{1+2a})} = \lim_{h\to 0} \frac{1}{(\sqrt{1+2(a+h)} + \sqrt{1+2a})} = \frac{1}{2\sqrt{1+2a}}$$

(b) $g(x) = \frac{1}{x^2}$

$$\lim_{h\to 0} \frac{\frac{1}{(a+h)^2} - \frac{1}{a^2}}{h} = \lim_{h\to 0} \frac{\frac{a^2-(a+h)^2}{a^2(a+h)^2}}{h} = \lim_{h\to 0} \frac{1}{h} \frac{a^2 - a^2 - 2ah - h^2}{a^2(a+h)^2}$$

$$\lim_{h\to 0} \frac{-2a-h}{a^2(a+h)^2} = -\frac{2}{a^3}$$

(c) $h(x) = x + \sqrt{x}$

$$\lim_{h\to 0} \frac{[a+h+\sqrt{a+h}] - [a+\sqrt{a}]}{h} = \lim_{h\to 0} \frac{h+\sqrt{a+h}-\sqrt{a}}{h} = \lim_{h\to 0} \left(1 + \frac{\sqrt{a+h}-\sqrt{a}}{h}\right) =$$

$$1 + \lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} \cdot \frac{\sqrt{a+h}+\sqrt{a}}{\sqrt{a+h}+\sqrt{a}} = 1 + \lim_{h\to 0} \frac{h}{h(\sqrt{a+h}+\sqrt{a})} = 1 + \frac{1}{2\sqrt{a}}$$

(d) $f(x) = \frac{2}{\sqrt{3-x}}$

$$\lim_{h\to 0} \frac{\frac{2}{\sqrt{3-(a+h)}} - \frac{2}{\sqrt{3-a}}}{h} = \lim_{h\to 0} \frac{2(\sqrt{3-a} - \sqrt{3-a-h})}{h\sqrt{3-a}\sqrt{3-a-h}} \cdot \frac{\sqrt{3-a}+\sqrt{3-a-h}}{\sqrt{3-x}+\sqrt{3-a-h}} =$$

$$\lim_{h\to 0} \frac{2}{\sqrt{3-a}\sqrt{3-a-h}(\sqrt{3-a}+\sqrt{3-a-h})} = \frac{2}{(3-a)2\sqrt{3-a})} = \frac{1}{(3-a)^{3/2}}$$

(e) $f(x) = \frac{x}{x^2-1}$

$$\lim_{h\to 0} \frac{\frac{a+h}{(a+h)^2-1} - \frac{a}{a^2-1}}{h} = \lim_{h\to 0} \frac{(a+h)(a^2-1) - a((a+h)^2-1)}{h((a+h)^2-1)(a^2-1)} = \lim_{h\to 0} \frac{-ha^2+ah^2-h}{h((a+h)^2-1)(a^2-1)} =$$

$$\frac{-a^2-1}{(a^2-1)^2}$$

19. A space traveler is moving from left to right along the curve $y = x^2$. When she shuts off the engines, she will go off along the tangent line at that point. At what point should she shut off the engines in order to reach the point $(4, 15)$? (Hint: Label the unknown point on the graph of $y = x^2$ as (a, a^2)).

An alternative way of stating this problem: Find the equation(s) of the tangent lines to $y = x^2$ that go through the additional point $(4, 15)$. If we find these values of x, since we're moving from left to right, we'd choose the smaller x.

Let a be the $x-$coordinate we're looking for. Then the line goes through the points (a, a^2) and $(4, 15)$. This says that the slope of the line should be:

$$m = \frac{a^2 - 15}{a - 4}$$

On the other hand, this is a tangent line to x^2, so the slope of the tangent line at $x = a$ should be the derivative, $2a$.

Putting these together,

$$2a = \frac{a^2 - 15}{a - 4}$$

Clear the fractions and simplify to get: $a^2 - 8a + 15 = 0$. The solutions are $a = 3, a = 5$. We choose the smaller value, $a = 3$, since we're moving from left to right.