Exam 2 Review: 2.8-3.6, Plus Applications From 3.7/3.8

This portion of the course covered the bulk of the formulas for differentiation, together with a few definitions and techniques. Remember that we also left 2.8 for this exam.

From 2.8, we should be able to plot the derivative given a graph of f, compute the derivative using the definition, and know the relationship between continuity and differentiability.

For Chapter 3, the following tables summarize the rules that we've had:

$f(x)$	$f'(x)$	Sect
c	0	3.1
x^n	nx^{n-1}	3.1
a^x	$a^x \ln(a)$	3.1
e^x	e^x	3.1
$\log_a(x)$	$\frac{1}{x\ln(a)}$	3.6
$\ln(x)$	$\frac{1}{x}$	3.6
$\sin(x)$	$\cos(x)$	3.3
$\cos(x)$	$-\sin(x)$	3.3
$\tan(x)$	$\sec^2(x)$	3.3
$\sec(x)$	$\sec(x)\tan(x)$	3.3
$\csc(x)$	$-\csc(x)\cot(x)$	3.3
$\cot(x)$	$-\csc^2(x)$	3.3
$\sin^{-1}(x)$	$\frac{1}{\sqrt{1-x^2}}$	3.5
$\tan^{-1}(x)$	$\frac{1}{1+x^2}$	3.5

$f(x)$	$f'(x)$	Sect
cf	cf'	3.1
$f \pm g$	$f' \pm g'$	3.1
$f \cdot g$	$f'g + fg'$	3.2
$\frac{f}{g}$	$\frac{f'g-fg'}{g^2}$	3.2
$f(g(x))$	$f'(g(x))g'(x)$	3.4
$f(x)^{g(x)}$	Logarithmic Diff	3.6
Eqn in x, y	Implicit Diff	3.5

Vocabulary/Techniques:

- Be sure you distinguish between:

$$a^x \text{ or } a^{f(x)} \qquad x^a \text{ or } (f(x))^a \qquad f(x)^{g(x)}$$

- Know the definition of "differentiable".
- Understand the relationship between "differentiable" and "continuous".
- Implicit Differentiation: A technique where we are given an equation with x, y. We treat y as a function of x, and differentiate without explicitly solving for y first.

  Example: $x^2y + \sqrt{xy} = 6x \to 2xy + x^2y' + \frac{1}{2}(xy)^{-\frac{1}{2}}(y + xy') = 6$
- Logarithmic Differentiation: A technique where we apply the logarithm to $y = f(x)$ before differentiating. Used for taking the derivative of complicated expressions, and needed for taking the derivative of $f(x)^{g(x)}$.

  Example: $y = x^x \to \ln(y) = x\ln(x) \to \frac{1}{y}y' = \ln(x) + 1 \to$ ... etc
- Differentiation of Inverses: If we know the derivative of $f(x)$, then we can determine the derivative of $f^{-1}(x)$. This technique was used to find derivatives of the inverse trig functions, for example:

$y = f^{-1}(x) \Rightarrow f(y) = x \Rightarrow f'(y)y' = 1$ From this, we could write:

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$$

Alternatively, we say that if (a, b) is on the graph of f and $f'(a) = m$, then we know that (b, a) is on the graph of f^{-1}, and $\frac{df^{-1}}{dx}(b) = \frac{1}{m}$.

NOTE: This is NOT the same as the derivative of $(f(x))^{-1} = \frac{1}{f(x)}$, which is

$$\frac{d}{dx}\left((f(x))^{-1}\right) = -(f(x))^{-2} f'(x) = \frac{-f'(x)}{(f(x))^2}$$

- We also have an alternative version of the Chain Rule that may be useful in certain cases.

  For example, if Volume is a function of Radius, Radius is a function of Pressure, Pressure is a function of time, then we can find the rate of change of Volume in terms of the Radius, or the Pressure, or the time. Respectively, this is:

  $$\frac{dV}{dR}, \qquad \frac{dV}{dP} = \frac{dV}{dR} \cdot \frac{dR}{dP}, \qquad \frac{dV}{dt} = \frac{dV}{dR} \cdot \frac{dR}{dP} \cdot \frac{dP}{dt}$$

  In fact, we could also find things like dR/dV, dP/dR, and so on because of the relationship between the derivative of a function and its inverse: $dx/dy = 1/(dy/dx)$.

- Things that come up in the inverse trig stuff: Be able to simplify expressions like $\tan(\cos^{-1}(x))$, $\sin(\tan^{-1}(x))$, etc. using an appropriate right triangle.

- Remember the logarithm rules:

  1. $A = \mathrm{e}^{\ln(A)}$ for any $A > 0$.
  2. $\log(ab) = \log(a) + \log(b)$
  3. $\log(a/b) = \log(a) - \log(b)$
  4. $\log(a^b) = b\log(a)$

- Always simplify BEFORE differentiating. Example: To differentiate $y = x\sqrt{x}$, first rewrite as $y = x^{3/2}$

Exam 2 Review Questions

Sections 2.8-3.6 will be on the exam, and we will include several applications from 3.7 and 3.8 that will be discussed in class. You should review this sheet, and don't forget to go over old quizzes and homework.

If you would like extra problems, the chapter reviews in our text are excellent. *Good Luck!*

1. Finish the definition:
   (a) The derivative of f is: $f'(x) =$
   (b) A function f is said to be *differentiable* at a point $x = a$ if:
   (c) A function f is said to be differentiable on the (open) interval (a, b) if:

2. Short Answer:
   (a) How do we define the inverse sine function? (Pay attention to the domain, range and whether the domain, range are angle measures or the ratios of a triangle).
   (b) What is a *normal line*?
   (c) How do we differentiate a function that involves the absolute value?
   (d) $\lim_{\theta\to 0} \frac{\sin(\theta)}{\theta}$ =? (You don't need to justify)
   (e) $\lim_{\theta\to 0} \frac{\cos(\theta)-1}{\theta}$ =? (You don't need to justify)
   (f) $\lim_{\theta\to 0} \frac{\tan(3t)}{\sin(2t)}$ =? (Do provide details)
   (g) If $f(x) = \sqrt{x}$, find a formula for $f'(x)$ *using the definition of the derivative.*
   (h) If $f(x) = 3/x$, find a formula for $f'(x)$ using the definition of the derivative.

3. Fill in the missing steps in the proof of the Quotient Rule:
   $\frac{d}{dx}\left(\frac{f}{g}\right)$ =??

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{??}{g(x)g(x+h)}$$

$$= \lim_{h\to 0} \frac{1}{g(x)g(x+h)} \frac{f(x+h)g(x) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x+h)}{h}$$

$$= \lim_{h\to 0} \frac{1}{g(x)g(x+h)} \left(g(x)\frac{f(x+h)-f(x)}{h} - f(x)\frac{g(x+h)-g(x)}{h} \right)$$

=??

4. True or False, and explain:
   (a) The derivative of a polynomial is a polynomial.
   (b) If f is differentiable, then $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$
   (c) The derivative of $y = \sec^{-1}(x)$ is the derivative of $y = \cos(x)$.

(d) $\frac{d}{dx}(10^x) = x10^{x-1}$

(e) If $y = \ln|x|$, then $y' = \frac{1}{x}$

(f) The equation of the tangent line to $y = x^2$ at $(1, 1)$ is:

$$y - 1 = 2x(x - 1)$$

(g) If $y = e^2$, then $y' = 2e$

(h) If $y = |x^2 - x|$, then $y' = |2x - 1|$.

(i) If $y = ax + b$, then $\frac{dy}{da} = x$

5. Find the equation of the tangent line to $x^3 + y^3 = 3xy$ at the point $(\frac{3}{2}, \frac{3}{2})$.

6. If $f(0) = 0$, and $f'(0) = 2$, find the derivative of $f(f(f(f(x))))$ at $x = 0$.

7. If $f(x) = 2x + e^x$, find the equation of the tangent line **to the inverse** of f at $(1, 0)$. HINT: Do not actually try to compute f^{-1}.

8. Derive the formula for the derivative of $y = \csc^{-1}(x)$ using implicit differentiation.

9. Find the equation of the tangent line to $\sqrt{y} + xy^2 = 5$ at the point $(4, 1)$.

10. If $s^2t + t^3 = 1$, find $\frac{dt}{ds}$ and $\frac{ds}{dt}$.

11. If $y = x^3 - 2$ and $x = 3z^2 + 5$, then find $\frac{dy}{dz}$.

12. A space traveler is moving from left to right along the curve $y = x^2$. When she shuts off the engines, she will go off along the tangent line at that point. At what point should she shut off the engines in order to reach the point $(4, 15)$?

13. A particle moves in the plane according to the law $x = t^2 + 2t$, $y = 2t^3 - 6t$. Find the slope of the tangent line when $t = 0$. HINT: We can say that $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$

14. Find h' in terms of f' and g', if: $h(x) = \dfrac{f(x)g(x)}{f(x) + g(x)}$

15. The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.

    (a) Find the rate of change of the volume with respect to the radius if the height is constant.

    (b) Find the rate of change of the volume with respect to time if both the height and the radius are functions of time.

16. Find the coordinates of the point on the curve $y = (x - 2)^2$ at which the tangent line is perpendicular to the line $2x - y + 2 = 0$.

17. For what value(s) of A, B, C does the polynomial $y = Ax^2+Bx+C$ satisfy the differential equation:

$$y'' + y' - 2y = x^2$$

Hint: If $c_1x^2 + c_2x + c_3 = x^2$ for all x, then $c_1 = 1, c_2 = 0, c_3 = 0$.

18. If $V = \sin(w)$, $w = \sqrt{u}$, $u = t^2 + 3t$, compute: The rate of change of V with respect to w, the rate of change of V with respect to u, and the rate of change of V with respect to t.

19. Find all value(s) of k so that $y = e^{kt}$ satisfies the differential equation:

$$y'' - y' - 2y = 0$$

20. Find the points on the ellipse $x^2 + 2y^2 = 1$ where the tangent line has slope 1.

21. Differentiate. You may assume that y is a function of x, if not already defined explicitly. If you use implicit differentiation, solve for $\frac{dy}{dx}$.

    (a) $y = \log_3(\sqrt{x} + 1)$
    (b) $\sqrt{2xy} + xy^3 = 5$
    (c) $y = \sqrt{x^2 + \sin(x)}$
    (d) $y = e^{\cos(x)} + \sin(5^x)$
    (e) $y = \cot(3x^2 + 5)$
    (f) $y = x^{\cos(x)}$
    (g) $y = \sqrt{\sin(\sqrt{x})}$
    (h) $\sqrt{x} + \sqrt[3]{y} = 1$
    (i) $x\tan(y) = y - 1$
    (j) $y = \sqrt{x}\, e^{x^2}(x^2 + 1)^{10}$ (Hint: Logarithmic Diff)
    (k) $y = \sin^{-1}\left(\tan^{-1}(x)\right)$
    (l) $y = \ln|\csc(3x) + \cot(3x)|$
    (m) $y = \frac{-2}{\sqrt[4]{t^3}}$
    (n) $y = x3^{-1/x}$
    (o) $y = x\tan^{-1}(\sqrt{x})$
    (p) $y = e^{2^{e^x}}$
    (q) Let a be a positive constant. $y = x^a + a^x$
    (r) $x^y = y^x$
    (s) $y = \ln\left(\sqrt{\frac{3x+2}{3x-2}}\right)$

22. Be sure you're comfortable with graphical exercises like the following:

    (a) (2.8) 3, 4-11, 37-40, 43-46
    (b) (3.2) 49-50
    (c) (3.4) 65-67 (and similarly, 63)