Calculus I Review Solutions

1. Finish the definition:

   (a) $\lim_{x \to a} f(x) = L$ means that, for every $\epsilon > 0$, there is a $\delta > 0$ such that

   $$\text{if} \quad 0 < |x - a| < \delta \text{ then} \quad |f(x) - L| < \epsilon$$

   (b) A function f is continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$

   (c) $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

2. What are the three "Value" Theorems (state each):

   - IVT: Let f be continuous on $[a, b]$. Let N be any number between $f(a)$ and $f(b)$. Then there is a c in $[a, b]$ such that $f(c) = N$.
   - EVT: Let f be continuous on $[a, b]$. Then f attains a global max and global min on $[a, b]$.
   - MVT: Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there is a c in (a, b) so that

   $$f'(c) = \frac{f(b) - f(a)}{b - a} \qquad \text{or equivalently} \qquad f(b) - f(a) = f'(c)(b - a)$$

3. Find the point on the parabola $x + y^2 = 0$ that is closest to the point $(0, -3)$.

   SOLUTION: Let $(-a^2, a)$ be a point on the parabola. Then using the **square** of the distance formula, the distance to the point $(0, -3)$ is:

   $$U = (0 - -a^2)^2 + (-3 - a)^2 = a^4 + (a + 3)^2$$

   This does not factor easily (I will give you only quadratics to factor on the exam), but in this case, it is possible to factor the expression as

   $$2(a + 1)(2a^2 - 2a + 3) = 0 \quad \Rightarrow \quad a = -1$$

   Check to see if this is a max or min:

U	$-$	$+$
	$a < -1$	$a > -1$

   Therefore, we have a global minimum at the point where $a = -1$, which corresponds to $(-1, 1)$ (and by symmetry) $(-1, -1)$.

4. Write the equation of the line tangent to

   $$x = \sin(2y)$$

   at $x = 1$.

SOLUTION: We can compute this directly using implicit differentiation:

$$1 = \cos(2y)2\frac{dy}{dx} \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{1}{2\cos(2y)}$$

At $x = 1$, then $1 = \sin(2y)$, or $y = \frac{1}{2}\sin^{-1}(1)$. The inverse sine of 1 is $\pi/2$, so $y = \frac{\pi}{4}$ and

$$\frac{dy}{dx} = \frac{1}{2\cos(\pi/2)} = \text{``}\frac{1}{0}\text{''}$$

So the tangent line in this case is a vertical line $x = 1$. *NOTE: It was by chance that I picked that point; When I ask for the equation of the tangent line, it will not normally be a vertical line.*

5. For what value(s) of A, B, C will $y = Ax^2 + Bx + C$ satisfy the differential equation

$$\frac{1}{2}y'' - 2y' + y = 3x^2 + 2x + 1$$

SOLUTION: Substitute y into the differential equation, then equate coefficients to $3x^2 + 2x + 1$:

$$\begin{array}{rlll} y = & Ax^2 & +Bx & +C \\ y' = & & +2Ax & +B \\ y'' = & & & 2A \end{array} \quad \Rightarrow \quad \begin{array}{rlll} y = & Ax^2 & +Bx & +C \\ -2y' = & & -4Ax & -2B \\ (1/2)y'' = & & & A \\ \hline 3x^2 + 2x + 1 = & Ax^2 & +(B-4A)x & +(A-2B+C) \end{array}$$

Therefore,

$$A = 3, \qquad B - 4A = 2 \qquad A - 2B + C = 1$$

Solving for these, we get $A = 3, B = 14$ and $C = 26$.

6. For what value(s) of k will $y = e^{kt}$ satisfy the differential equation

$$y'' + 4y' + 3y = 0$$

SOLUTION: Substituting the function $y = e^{kt}$, $y' = ke^{kt}$, and $y'' = k^2e^{kt}$:

$$k^2e^{kt} + 4ke^{kt} + 3e^{kt} = 0 \quad \Rightarrow \quad e^{kt}(k^2 + 4k + 3) = 0 \quad \Rightarrow \quad e^{kt}(k+1)(k+3) = 0$$

Therefore, $k = -1$ or $k = -3$.

7. Compute the derivative of y with respect to x:

   (a) $y = \sqrt[3]{2x+1}\sqrt[5]{3x-2}$
   SOLUTION: Use logarithmic differentiation to get:

$$\ln(y) = \frac{1}{3}\ln(2x+1) + \frac{1}{5}\ln(3x-2)$$

so that

$$\frac{1}{y}y' = \frac{1}{3}\cdot\frac{1}{2x+1}\cdot 2 + \frac{1}{5}\cdot\frac{1}{3x-2}\cdot 3$$

Now,

$$y' = \left(\sqrt[3]{2x+1}\sqrt[5]{3x-2}\right)\left(\frac{2}{3(2x+1)} + \frac{3}{5(3x-2)}\right)$$

(b) $y = \frac{1}{1+u^2}$, where $u = \frac{1}{1+x^2}$

SOLUTION: In this case, $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$, so

$$\frac{dy}{du} = \frac{-2u}{(1+u^2)^2}, \quad \frac{du}{dx} = \frac{-2x}{(1+x^2)^2}$$

so

$$\frac{dy}{dx} = \frac{4ux}{(1+u^2)^2(1+x^2)^2}$$

with $u = \frac{1}{1+x^2}$, which you can either state or explicitly substitute.

(c) $\sqrt[3]{y} + \sqrt[3]{x} = 4xy$

SOLUTION: Use implicit differentiation to get:

$$\frac{1}{3}y^{-2/3}\frac{dy}{dx} + \frac{1}{3}x^{-2/3} = 4y + 4x\frac{dy}{dx}$$

Bring all the $\frac{dy}{dx}$ terms together:

$$\left(\frac{1}{3}y^{-2/3} - 4x\right)\frac{dy}{dx} = 4y - \frac{1}{3}x^{-2/3} \Rightarrow \frac{dy}{dx} = \frac{4y - \frac{1}{3}x^{-2/3}}{\frac{1}{3}y^{-2/3} - 4x}$$

(d) $\sqrt{x+y} = \sqrt[3]{x-y}$

SOLUTION:Another implicit differentation:

$$\frac{1}{2}(x+y)^{-1/2}(1+y') = \frac{1}{3}(x-y)^{-2/3}(1-y')$$

Multiply out so that we can isolate y'

$$\frac{1}{2}(x+y)^{-1/2} + y'\cdot\frac{1}{2}(x+y)^{-1/2} = \frac{1}{3}(x-y)^{-2/3} - y'\cdot\frac{1}{3}(x-y)^{-2/3}$$

Now isolate y'

$$y'\left(\frac{1}{2}(x+y)^{-1/2} + \frac{1}{3}(x-y)^{-2/3}\right) = \frac{1}{3}(x-y)^{-2/3} - \frac{1}{2}(x+y)^{-1/2}$$

Final answer:

$$y' = \frac{\frac{1}{3}(x-y)^{-2/3} - \frac{1}{2}(x+y)^{-1/2}}{\frac{1}{2}(x+y)^{-1/2} + \frac{1}{3}(x-y)^{-2/3}}$$

(e) $y = \sin(2\cos(3x))$

SOLUTION:

$$\frac{dy}{dx} = \cos(2\cos(3x))(-2)\sin(3x)3 = -6\cos(2\cos(3x))\sin(3x)$$

(f) $y = (\cos(x))^{2x}$

SOLUTION: Use logarithmic differentiation for $f(x)^{g(x)}$ form:

$$\ln(y) = 2x\ln(\cos(x)) \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = 2\ln(\cos(x)) + 2x\frac{1}{\cos(x)}(-\sin(x))$$

$$\frac{dy}{dx} = (\cos(x))^{2x}\left(2\ln(\cos(x)) - 2x\tan(x)\right)$$

(g) $y = \sin^{-1}(3x) + 4^{3x} + \frac{x}{x+2}$

SOLUTION:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1+(3x)^2}} \cdot 3 + 4^{3x}\ln(4)\cdot 3 + \frac{1(x+2) - x(1)}{(x+2)^2}$$

$$\frac{dy}{dx} = \frac{3}{\sqrt{1+9x^2}} + 4^{3x}\ln(4)\cdot 3 + \frac{2}{(x+2)^2}$$

(h) $y = \log_3(x^2 - x)$

SOLUTION: (Rewrite the expression first)

$$3^y = x^2 - x \quad \Rightarrow \quad 3^y\ln(3)\frac{dy}{dx} = 2x - 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2x-1}{3^y\ln(3)} = \frac{2x-1}{(x^2-x)\ln(3)}$$

(i) $y = \cot(3x^2 + 5)$

SOLUTION: $y' = -csc^2(3x^2+5)(6x)$

(j) $\sqrt{x} + \sqrt[3]{y} = 1$

SOLUTION: Implicit Differentiation:

$$\frac{1}{2}x^{-1/2} + \frac{1}{3}y^{-2/3}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{3y^{2/3}}{2x^{1/2}}$$

(k) $x\tan(y) = y - 1$

Product rule/Implicit Diff

$$\tan(y) + x\sec^2(y)y' = y' \Rightarrow \tan(y) = y'(1 - x\sec^2(y))$$

Solve for y':

$$y' = \frac{\tan(y)}{1 - x\sec^2(y)}$$

(l) $y = \frac{-2}{\sqrt[4]{t^3}}$, where $t = \ln(x^2)$.

First, note that $y = -2t^{-3/4}$ and $t = 2\ln(x)$. Now,

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

where

$$\frac{dy}{dt} = -2 \cdot \frac{-3}{4} t^{-7/4} = \frac{3}{2} t^{-7/4}, \qquad \frac{dt}{dx} = \frac{2}{x}$$

so put it all together (and substitute back for x):

$$\frac{dy}{dx} = \frac{3}{2} \left(2\ln(x)\right)^{-7/4} \cdot \frac{2}{x} = \frac{3}{x(2\ln(x))^{7/4}}$$

(m) $y = 3^{-1/x}$

SOLUTION: $y' = 3^{-1/x} \ln(3) \cdot (1/x^2)$

8. Find the local maximums and minimums: $f(x) = x^3 - 3x + 1$ Show your answer is correct by using both the first derivative test and the second derivative test.

SOLUTION: To find local maxs and mins, first differentiate to find critical points:

$$f'(x) = 3x^2 - 3 = 0 \Rightarrow x = \pm 1$$

For the first derivative test, set up a sign chart. You should see that $3x^2 - 3 = 3(x+1)(x-1)$ is positive for $x < 1$ and $x > 1$, and $f'(x)$ is negative if $-1 < x < 1$. Therefore, at $x = -1$, the derivative changes sign from positive to negative, so $x = -1$ is the location of a local maximum. At $x = 1$, the derivative changes sign from negative to positive, so we have a local minimum.

For the second derivative test, we compute the second derivative at the critical points:

$$f''(x) = 6x$$

so at $x = -1$, f is concave down, so we have a local max, and at $x = 1$, f is concave up, so we have a local min.

9. Compute the limit, if it exists. You may use any method (except a numerical table).

(a) $\lim_{x \to 0} \frac{x - \sin(x)}{x^3}$

We have a form of $\frac{0}{0}$, so use L'Hospital's rule:

$$\lim_{x \to 0} \frac{x - \sin(x)}{x^3} = \lim_{x \to 0} \frac{1 - \cos(x)}{3x^2}$$

We still have $\frac{0}{0}$, so do it again and again!

$$\lim_{x \to 0} \frac{1 - \cos(x)}{3x^2} = \lim_{x \to 0} \frac{\sin(x)}{6x} = \lim_{x \to 0} \frac{\cos(x)}{6} = \frac{1}{6}$$

(b) $\lim_{x \to 0} \dfrac{1 - e^{-2x}}{\sec(x)}$

Note that $\sec(0) = \frac{1}{\cos(0)} = 1$, so this function is continuous at $x = 0$ (we can substitute $x = 0$ in directly), and we get that the limit is 0.

(c) $\lim_{x \to 4^+} \dfrac{x-4}{|x-4|}$

Rewrite the expression to get rid of the absolute value:

$$\frac{x-4}{|x-4|} = \begin{cases} \frac{x-4}{x-4} & \text{if } x > 4 \\ \frac{x-4}{-(x-4)}, & \text{if } x < 4 \end{cases} = \begin{cases} 1 & \text{if } x > 4 \\ -1 & \text{if } x < 4 \end{cases}$$

Therefore,

$$\lim_{x \to 4^+} \frac{x-4}{|x-4|} = 1$$

(Note that the overall limit does not exist, however).

(d) $\lim_{x \to -\infty} \sqrt{\dfrac{2x^2 - 1}{x + 8x^2}}$

For this problem, we should recall that if $x < 0$, then $x = -\sqrt{x^2}$, although in this particular case, the negative signs will cancel:

$$\lim_{x \to -\infty} \sqrt{\frac{2x^2 - 1}{x + 8x^2}} \cdot \frac{\frac{-1}{\sqrt{x^2}}}{\frac{-1}{\sqrt{x^2}}} = \lim_{x \to -\infty} \sqrt{\frac{2 - \frac{1}{x^2}}{\frac{1}{x} + 8}} = \sqrt{\frac{2}{8}} = \frac{1}{2}$$

(e) $\lim_{x \to \infty} \sqrt{x^2 + x + 1} - \sqrt{x^2 - x}$

Multiply by the conjugate (or rationalize):

$$\lim_{x \to \infty} \sqrt{x^2 + x + 1} - \sqrt{x^2 - x} \cdot \frac{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} = \lim_{x \to \infty} \frac{2x + 1}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}}.$$

Now divide numerator and denominator:

$$\lim_{x \to \infty} \frac{2x + 1}{\sqrt{x^2 + x + 1} + \sqrt{x^2 - x}} \cdot \frac{\frac{1}{x}}{\frac{1}{\sqrt{x^2}}} = \lim_{x \to \infty} \frac{2 + \frac{1}{x}}{\sqrt{1 + \frac{1}{x} + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x}}} = 1$$

(f) $\lim_{h \to 0} \dfrac{(1+h)^{-2} - 1}{h}$

For practice, we'll try it without using L'Hospital's rule:

$$\lim_{h \to 0} \frac{(1+h)^{-2} - 1}{h} = \lim_{h \to 0} \frac{\frac{1}{(1+h)^2} - 1}{h} = \lim_{h \to 0} \frac{1 - (1+h)^2}{h(1+h)^2} = \lim_{h \to 0} \frac{-2h - h^2}{h(1+h)^2} = -2$$

With L'Hospital:

$$\lim_{h \to 0} \frac{(1+h)^{-2} - 1}{h} = \lim_{h \to 0} \frac{-2(1+h)^{-3}}{1} = -2$$

EXTRA: This limit was the derivative of some function at some value of x. Name the function and the x value[1].

[1] The function is $f(x) = x^{-2}$ at $x = 1$

(g) $\lim_{x\to\infty} x^3 e^{-x^2}$

First rewrite the function so that it's in an acceptable form for L'Hospital's rule:

$$\lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \lim_{x\to\infty} \frac{3x^2}{2xe^{x^2}}$$

Note that $\frac{3x^2}{2xe^{x^2}} = \frac{3x}{2e^{x^2}}$, and again use L'Hospital's rule:

$$\lim_{x\to\infty} \frac{3x}{2e^{x^2}} = \lim_{x\to\infty} \frac{3}{4xe^{x^2}} = 0$$

(h) $\lim_{x\to 1} \frac{x^{1000}-1}{x-1}$

Using L'Hospital's rule:

$$\lim_{x\to 1} \frac{x^{1000}-1}{x-1} = \lim_{x\to 1} \frac{1000x^{999}}{1} = 1000$$

(i) $\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)}$

Recall that $\tan^{-1}(0) = 0$, since $\tan(0) = 0$, so this is in a form for L'Hospital's rule:

$$\lim_{x\to 0} \frac{x}{\tan^{-1}(4x)} = \lim_{x\to 0} \frac{1}{\frac{1}{1+(4x)^2}\cdot 4} = \lim_{x\to 0} \frac{1+16x^2}{4} = \frac{1}{4}$$

(j) $\lim_{x\to 1} x^{\frac{1}{1-x}}$

In this case, recall that $x^{\frac{1}{1-x}} = e^{\frac{1}{1-x}\cdot \ln(x)}$, so:

$$\lim_{x\to 1} x^{\frac{1}{1-x}} = \lim_{x\to 1} e^{\frac{1}{1-x}\cdot \ln(x)} = e^{\lim_{x\to 1} \frac{\ln(x)}{1-x}}$$

so we focus on the exponent:

$$\lim_{x\to 1} \frac{\ln(x)}{1-x} = \lim_{x\to 1} \frac{\frac{1}{x}}{-1} = -1$$

so the overall limit:

$$\lim_{x\to 1} x^{\frac{1}{1-x}} = e^{-1}$$

10. Determine all vertical/horizontal asymptotes and critical points of $f(x) = \frac{2x^2}{x^2-x-2}$

The vertical asymptotes: $x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$, so $x = -1, x = 2$ are the equations of the vertical asymptotes (note that the numerator is not zero at these values).

The horizontal asymptotes:

$$\lim_{x\to\pm\infty} \frac{2x^2}{x^2-x-2} = \lim_{x\to\pm\infty} \frac{2}{1-\frac{1}{x}-\frac{2}{x^2}} = 2$$

so $y = 2$ is the horizontal asymptote (for both $+\infty$ and $-\infty$).

For the critical points, you should find that the numerator of the derivative simplifies to $-2x(x+4)$, so the critical points are $x = 0$ and $x = -4$.

11. Find values of m and b so that (1) f is continuous, and (2) f is differentiable.

$$f(x) = \begin{cases} x^2 & \text{if} \quad x \leq 2 \\ mx + b & \text{if} \quad x > 2 \end{cases}$$

First we see that if $x < 2$, $f(x) = x^2$ which is continuous, and if $x < 2$, $f(x) = mx + b$, which is also continuous for any value of m and b. The only problem point is $x = 2$, so we check the three conditions from the definition of continuity:

- $f(2) = 4$, so $f(2)$ exists.
- To compute the limit, we have to do them separately:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 4$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} mx + b = 2m + b$$

  For the limit to exist, we must have $4 = 2m + b$. This will also automatically make item 3 true.

  There are an infinite number of possible solutions. Given any m, $b = 4 - 2m$.

For the second part, we know that f must be continuous to be differentiable, so that leaves us with $b = 4 - 2m$. Also, the derivatives need to match up at $x = 2$. On the left side of $x = 2$, $f'(x) = 2x$ and on the right side of $x = 2$, $f'(x) = m$. Therefore, $4 = m$ and $b = 4 - 2 \cdot 4 = -4$.

To be differentiable at $x = 2$, we require $m = 4$ and $b = -4$.

12. Find the local and global extreme values of $f(x) = \frac{x}{x^2+x+1}$ on the interval $[-2, 0]$.

We see that $x^2 + x + 1 = 0$ has no solution, so $f(x)$ is continuous on $[-2, 0]$. Therefore, the extreme value theorem is valid. Next, find the critical points:

$$f'(x) = \frac{(x^2 + x + 1) - x(2x + 1)}{(x^2 + x + 1)^2} = \frac{-x^2 + 1}{(x^2 + x + 1)^2}$$

so the critical points are $x = \pm 1$ of which we are only concerned with $x = -1$. Now build a chart of values:

x	0	-1	-2
$f(x)$	0	-1	$-2/3$

The minimum occurs at $x = -1$ and the maximum occurs at $x = 0$. The minimum value is -1 and the maximum value is 0.

For the local min/max, use a sign chart for $f'(x)$ in the interval $[-2, 0]$ (or you can use the second derivative test). We see that the denominator of $f'(x)$ is always positive, and the numerator is $1 - x^2$, which changes from negative to positive at $x = -1$, so that f is decreasing then increasing and there is a local minimum at $x = -1$.

13. Suppose f is differentiable so that:

$$f(1) = 1,\ f(2) = 2,\ f'(1) = 1\ f'(2) = 2$$

If $g(x) = f(x^3 + f(x^2))$, evaluate $g'(1)$.

Use the chain rule to get that:

$$g'(x) = f'\left(x^3 + f(x^2)\right) \cdot \left(3x^2 + f'(x^2) \cdot 2x\right)$$

Be careful with the parentheses!:

$$g'(1) = f'(1 + f(1)) \cdot (3 + 2f'(1)) = f'(1+1)(3+2) = 5f'(2) = 5 \cdot 2 = 10$$

14. Let $x^2y + a^2xy + \lambda y^2 = 0$

(a) Let a and λ be constants, and let y be a function of x. Calculate $\frac{dy}{dx}$:

$$2xy + x^2\frac{dy}{dx} + a^2y + a^2x\frac{dy}{dx} + 2\lambda y\frac{dy}{dx} = 0$$

$$(x^2 + a^2x + 2\lambda y)\frac{dy}{dx} = -(2xy + a^2y) \Rightarrow \frac{dy}{dx} = \frac{-(2xy + a^2y)}{x^2 + a^2x + 2\lambda y}$$

(b) Let x and y be constants, and let a be a function of λ. Calculate $\frac{da}{d\lambda}$:

$$2axy\frac{da}{d\lambda} + y^2 = 0 \Rightarrow \frac{da}{d\lambda} = \frac{-y^2}{2axy}$$

EXTRA[2]: What is $\frac{d\lambda}{da}$?

15. Show that $x^4 + 4x + c = 0$ has at most one solution in the interval $[-1, 1]$.

We don't need the Intermediate Value Theorem here, only the Mean Value Theorem. The derivative is $4x^3 + 4$, so the only critical point is $x = -1$, which is also an endpoint. This implies: (1) If $x^4 + 4x + c = 0$ had two solutions (which is possible), then one of them must be outside the interval, since the two solutions must be on either side of $x = -1$. Therefore, there could be one solution inside the interval. (2) There cannot be any other solution to $x^4 + 4x + c = 0$ inside the interval, because then there would have to be another critical point in $[-1, 1]$. Therefore, we conclude that there is at most one solution inside the interval (there might be no solutions).

16. True or False, and give a short explanation.

(a) If $f(x)$ is decreasing and $g(x)$ is decreasing, then $f(x)g(x)$ is decreasing.
FALSE: Since $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$, if f is decreasing we know that f' is negative (but $g(x)$ could be either positive or negative), and g is decreasing, so $g'(x)$ is negative (but f could be either positive or negative.

[2]The answer is $\frac{d\lambda}{da} = \frac{2axy}{-y^2}$

(b) $\frac{d}{dx}(10^x) = x10^{x-1}$
FALSE: This is a mis-application of the power rule. The derivative should use the exponential rule, $y' = 10^x \ln(10)$.

(c) If $f'(x)$ exists and is nonzero for all x, then $f(1) \neq f(0)$.
TRUE. If $f'(x)$ exists for all x, then f is differentiable everywhere (and so f is also continuous everywhere). Thus, the Mean Value Theorem applies. If $f(1) = f(0)$, that would imply the existence of a c in the interval $(0, 1)$ so that

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = 0$$

but we're told that $f'(x) \neq 0$.

(d) If $2x + 1 \leq f(x) \leq x^2 + 2$ for all x, then $\lim_{x\to 1} f(x) = 3$.
TRUE. This is the Squeeze Theorem. If $f(x)$ is trapped between $2x+1$ and x^2+2 for all x, and since the limit as $x \to 1$ of $2x + 1$ is 3, and the limit as $x \to 1$ of $x^2 + 2 = 3$, then that forces the limit as $x \to 1$ of $f(x)$ to also be 3.

(e) If $f'(r)$ exists, then $\lim\limits_{x\to r} f(x) = f(r)$
TRUE: This says that if f is differentiable at $x = r$, then f is continuous at $x = r$.

(f) If f and g are differentiable, then: $\frac{d}{\text{dx}}(f(g(x)) = f'(x)g'(x)$
FALSE: We need to use the chain rule to differentiate a composition.

(g) If $f(x) = x^2$, then the equation of the tangent line at $x = 3$ is: $y - 9 = 2x(x - 3)$
FALSE: The slope is 6, not $2x$.

(h) $\lim\limits_{\theta\to\frac{\pi}{3}} \dfrac{\cos(\theta) - \frac{1}{2}}{\theta - \frac{\pi}{3}} = -\sin\left(\dfrac{\pi}{3}\right)$
TRUE, by l'Hospital's rule. You could have also said that the expression is in the form:

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

with $f(x) = \cos(x)$ and $a = \frac{\pi}{3}$. This is another way of defining the derivative of $\cos(x)$ at $x = \frac{\pi}{3}$.

(i) $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$
FALSE, with the usual restrictions on the sine function. That is, if $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, then it is true that $\sin^{-1}(\sin(\theta)) = \theta$. In this case,

$$\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right) = \frac{\pi}{3}$$

(j) $5^{\log_5(2x)} = 2x$, for $x > 0$.
TRUE, since 5^x and $\log_5(x)$ are inverses of each other. We needed $x > 0$ so that $\log_5(2x)$ is defined.

(k) $\frac{d}{dx}\ln(|x|) = \frac{1}{x}$, for all $x \neq 0$.
TRUE: Compute it-

$$\ln|x| = \begin{cases} \ln(x), & \text{if } x > 0 \\ \ln(-x), & \text{if } x < 0 \end{cases}$$

so

$$\frac{d}{dx}\ln|x| = \begin{cases} \frac{1}{x}, & \text{if } x > 0 \\ \frac{1}{-x}\cdot(-1) = \frac{1}{x}, & \text{if } x < 0 \end{cases}$$

(l) $\lim_{x\to 1}\dfrac{x^2+6x-7}{x^2+5x-6} = \dfrac{\lim_{x\to 1} x^2+6x-7}{\lim_{x\to 1} x^2+5x-6}$
FALSE. The limit law does not apply if the limit in the denominator is zero (otherwise, it would have been true).

(m) If neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a} f(x)g(x)$ does not exist.
FALSE: It is possible for neither limit to exist, but the product does. For example,

$$f(x) = \begin{cases} 1 \text{ if } x \leq 0 \\ 0 \text{ if } x > 0 \end{cases} \qquad g(x) = \begin{cases} 0 \text{ if } x \leq 0 \\ 1 \text{ if } x > 0 \end{cases}$$

Then both f and g have a jump at $x = 0$. But, if you multiply them together, you get: $f(x)g(x) = 0$ for all x, which is continuous everywhere. (Main point: The conditions for the limit laws to hold were that each limit needs to exist separately before we can guarantee that the product, sum or difference of the limits also exists).

(n) $\frac{x^2-1}{x-1} = x + 1$
FALSE: It is true for every point except $x = 1$, which is not in the domain of the fraction, but is in the domain of $x + 1$.

17. Find the domain of $\ln(x - x^2)$:

Use a sign chart to determine where $x - x^2 = x(1-x) > 0$:

x	$-$	$+$	$+$
$1-x$	$+$	$+$	$-$
	$x<0$	$0<x<1$	$x>1$

so overall, $x - x^2 > 0$ if $0 < x < 1$.

18. Find the value of c guaranteed by the Mean Value Theorem, if $f(x) = \frac{x}{x+2}$ on the interval $[1, 4]$.

To set things up, we see that f is continuous on $[1, 4]$ and differentiable on $(1, 4)$, since the only "bad point" is $x = -2$. We should get that $f'(x) = \frac{2}{(x+2)^2}$, $f(1) = \frac{1}{3}$ and $f(4) = \frac{2}{3}$. Therefore, the Mean Value Theorem says that c should satisfy:

$$\frac{2}{(c+2)^2} = \frac{\frac{2}{3} - \frac{1}{3}}{4-1} = \frac{1}{9}$$

or

$$(c+2)^2 = 18 \Rightarrow c = -2 \pm \sqrt{18}$$

of which only $-2 + \sqrt{18} \approx 2.243$ is inside our interval.

19. Given that the graph of f passes through the point $(1, 6)$ and the slope of the tangent line at $(x, f(x))$ is $2x + 1$, find $f(2)$.

    Since $f'(x) = 2x + 1$, $f(x) = x^2 + x + C$ is the general antiderivative. Given that $(1, 6)$ goes through f, $1^2 + 1 + C = 6 \Rightarrow C = 4$. Therefore, $f(x) = x^2 + x + 4$. Now, $f(2) = 4 + 2 + 4 = 10$.

20. Simplify, using a triangle: $\cos(\tan^{-1}(x))$ (No calculus needed).

    SOLUTION: Draw a triangle with $\theta = \tan^{-1}(x)$ or $\tan(\theta) = x/1$, so x should be opposite θ and 1 is the length adjacent. The hypotenuse is then $\sqrt{1 + x^2}$, so that the cosine of θ is

$$\cos(\theta) = \frac{1}{\sqrt{1 + x^2}}$$

21. A fly is crawling from left to right along the curve $y = 8 - x^2$, and a spider is sitting at $(4, 0)$. At what point along the curve does the spider first see the fly?

    Another way to say this: What are the tangent lines through $y = 8 - x^2$ that also go through $(4, 0)$?

    The unknown value here is the $x-$coordinate, so let $x = a$. Then the slope is $-2a$, and the corresponding point on the curve is $(a, 8 - a^2)$. The general form of the equation of the tangent line is then given by:

$$y - 8 + a^2 = -2a(x - a)$$

where x, y are points on the tangent line. We want the tangent line to go through $(4, 0)$, so we put this point in and solve for a:

$$-8 + a^2 = -2a(4 - a) = -8a + 2a^2 \quad \Rightarrow \quad 0 = a^2 - 8a + 8$$

$$\Rightarrow \quad a = \frac{8 \pm \sqrt{32}}{2}$$

so we take the leftmost value, $a = \frac{8-\sqrt{32}}{2}$.

22. Compute the limit, without using L'Hospital's Rule. $\lim_{x \to 7} \frac{\sqrt{x + 2} - 3}{x - 7}$

    Rationalize to get:

$$\lim_{x \to 7} \frac{\sqrt{x + 2} - 3}{x - 7} \cdot \frac{\sqrt{x + 2} + 3}{\sqrt{x + 2} + 3} = \lim_{x \to 7} \frac{x + 2 - 9}{(x - 7)(\sqrt{x + 2} + 3)}$$

$$= \lim_{x \to 7} \frac{1}{(\sqrt{x + 2} + 3)} = \frac{1}{6}$$

which is the derivative of $\sqrt{x + 2}$ at $x = 7$.

23. For what value(s) of c does $f(x) = cx^4 - 2x^2 + 1$ have both a local maximum and a local minimum?

First, $f'(x) = 4cx^3 - 4x = 4x(cx^2 - 1)$, and $f''(x) = 12cx^2 - 4$. The candidates for the location of the local max's and min's are where $f'(x) = 0$, which are $x = 0$ and $x = \pm\sqrt{1/c}$ $(c > 0)$. We can use the second derivative test to check these out:

At $x = 0$, $f''(0) = -4$, so $x = 0$ is always a local max. At $x = \pm\sqrt{1/c}$, $f''(\pm\sqrt{1/c}) = 12 - 4 = 8$. So, if $c > 0$, there are local mins at $x = \pm\sqrt{1/c}$.

24. Find constants a and b so that $(1, 6)$ is an inflection point for $y = x^3 + ax^2 + bx + 1$.

Hint: The IVT might come in handy

SOLUTION: Differentiate twice to get:

$$y'' = 6x + 2a$$

At $x = 1$, we want an inflection point, so $6 + 2a$ should be a point where y'' changes sign: $6 + 2a = 0 \Rightarrow a = -3$. We see that if $a < -3$, then $y'' < 0$, and if $a > -3$, $y'' > 0$.

Putting this back into the function, we have:

$$y = x^3 - 3x^2 + bx + 1$$

and we want the curve to go through the point $(1, 6)$:

$$6 = 1 - 3 + b + 1$$

so $b = 7$.

25. Suppose that $F(x) = f(g(x))$ and $g(3) = 6$, $g'(3) = 4$, $f(3) = 2$ and $f'(6) = 7$. Find $F'(3)$.

By the Chain Rule:

$$F'(3) = f'(g(3))g'(3)$$

so $F'(3) = f'(6) \cdot 4 = 7 \cdot 4 = 28$

26. Find the dimensions of the rectangle of largest area that has its base on the $x-$axis and the other two vertices on the parabola $y = 8 - x^2$.

Try drawing a picture first: The parabola opens down, goes through the $y-$intercept at 8, and has $x-$intercepts of $\pm\sqrt{8}$.

Now, let x be as usual, so that the full length of the base of the rectangle is $2x$. Then the height is y, or $8 - x^2$. Therefore, the area of the rectangle is:

$$A = 2xy = 2x(8 - x^2) = 16x - 2x^3$$

and $0 \le x \le \sqrt{8}$. We see that the area will be zero at the endpoints, so we expect a maximum at the critical point inside the interval:

$$\frac{dA}{dx} = 16 - 6x^2$$

so the critical points are $x = \pm\frac{4}{\sqrt{6}}$, of which only $x = \frac{4}{\sqrt{6}}$ is in our interval. So the dimensions of the rectangle are:

$$2x = \frac{8}{\sqrt{6}}, y = 5\frac{1}{3}$$

27. Let $G(x) = h(\sqrt{x})$. Then where is G differentiable? Find $G'(x)$.

    First compute $G'(x) = h'(\sqrt{x})\frac{1}{2}x^{-1/2}$. From this we see that as long as h is differentiable and $x > 0$, then G will be differentiable.

28. If position is given by: $f(t) = t^4 - 2t^3 + 2$, find the times when the acceleration is zero. Then compute the velocity at these times.

    Take the second derivative, and set it equal to zero:

    $$f'(x) = 4t^3 - 6t^2, \quad f''(t) = 12t^2 - 12t = 0 \Rightarrow t = 0, t = 1$$

    The velocity at $t = 0$ is 0 and the velocity at $t = 1$ is $4 - 6 = -2$.

29. If $y = \sqrt{5t - 1}$, compute y'''.

    Nothing tricky here- Just differentiate, and differentiate, and differentiate!

    $$y' = \frac{1}{2}(5t-1)^{-1/2} \cdot 5 = \frac{5}{2}(5t-1)^{-1/2}$$
    $$y'' = \frac{5}{2} \cdot \frac{-1}{2}(5t-1)^{-3/2} \cdot 5 = \frac{-25}{4}(5t-1)^{-3/2}$$
    $$y''' = \frac{375}{8}(5t-1)^{-5/2}$$

30. If $f(x) = (2 - 3x)^{-1/2}$, find $f(0)$, $f'(0)$, $f''(0)$.

    $$f'(x) = \frac{3}{2}(2-3x)^{-3/2} \qquad f''(0) = \frac{27}{4}(2-3x)^{-52}$$

    then substitute $x = 0$ for each to get

    $$\frac{1}{\sqrt{2}} \qquad 3 \cdot 2^{-5/2} \qquad 27 \cdot 2^{-9/2}$$

31. Car A is traveling west at 50 mi/h, and car B is traveling north at 60 mi/h. Both are headed for the intersection between the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

    Let $A(t)$, $B(t)$ be the positions of cars A and B at time t. Let the distance between them be $z(t)$, so that the Pythagorean Theorem gives:

    $$z^2 = A^2 + B^2$$

Translating the question, we get that we want to find $\frac{dz}{dt}$ when $A = 0.3$, $B = 0.4$, (so $z = 0.5$), $A'(t) = 50$, $B'(t) = 60$. Then:

$$2z\frac{dz}{dt} = 2A\frac{dA}{dt} + 2B\frac{dB}{dt}$$

The two's divide out and put in the numbers:

$$0.5 \cdot \frac{dz}{dt} = 0.3 \cdot 50 + 0.4 \cdot 60$$

and solve for $\frac{dz}{dt}$, 78.

32. Find the linearization of $f(x) = \sqrt{1-x}$ at $x = 0$.

    SOLUTION: To compute the linearization, we find the equation of the tangent line. We should find that $f'(0) = -1/2$, and $f(0) = 1$, so

$$y - 1 = -\frac{1}{2}(x - 0) \quad \Rightarrow \quad y = -\frac{x}{2} + 1$$

33. Find $f(x)$, if $f''(x) = t + \sqrt{t}$, and $f(1) = 1$, $f'(1) = 2$.

$$f'(t) = \frac{1}{2}t^2 + \frac{2}{3}t^{3/2} + C$$

so $f'(1) = 2$ means:

$$\frac{1}{2} + \frac{2}{3} + C = 2, \text{ so } C = \frac{5}{6}$$

Now, $f'(t) = \frac{1}{2}t^2 + \frac{2}{3}t^{3/2} + \frac{5}{6}$, and

$$f(t) = \frac{1}{2} \cdot \frac{1}{3}t^3 + \frac{2}{3} \cdot \frac{2}{5}t^{5/2} + \frac{5}{6}t + C = \frac{1}{6}t^3 + \frac{4}{15}t^{5/2} + \frac{5}{6}t + C$$

Now, $f(1) = 1$ means:

$$\frac{1}{6} + \frac{4}{15} + \frac{5}{6} + C = 1 \Rightarrow \frac{5+8+25}{30} + C = 1 \Rightarrow C = 1 - \frac{19}{15} = \frac{-4}{15}$$

34. Find $f'(x)$ directly from the definition of the derivative (using limits and without L'Hospital's rule):

    (a) $f(x) = \sqrt{3-5x}$

$$\lim_{h\to 0} \frac{\sqrt{3-5x-5h} - \sqrt{3-5x}}{h} \cdot \frac{\sqrt{3-5x-5h} + \sqrt{3-5x}}{\sqrt{3-5x-5h} + \sqrt{3-5x}}$$

$$\lim_{h\to 0} \frac{3-5x-5h-3+5x}{h\left(\sqrt{3-5x-5h} + \sqrt{3-5x}\right)}$$

$$\frac{-5}{2\sqrt{3-5x}}$$

(b) $f(x) = x^2$

$$\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} =$$

$$\lim_{h\to 0} \frac{h(2x+h)}{h} = 2x$$

(c) $f(x) = x^{-1}$

$$\lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} =$$

$$\lim_{h\to 0} \frac{-h}{hx(x+h)} = \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2}$$

35. If $f(0) = 0$, and $f'(0) = 2$, find the derivative of $f(f(f(f(x))))$ at $x = 0$.

First, note that the derivative is (Chain Rule):

$$f'(f(f(f(x)))) \cdot f'(f(f(x))) \cdot f'(f(x)) \cdot f'(x)$$

which simplifies (since $f(0) = 0$) to:

$$f'(0) \cdot f'(0) \cdot f'(0) \cdot f'(0) = 2^4 = 16$$

36. Differentiate:

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0 \end{cases}$$

Is f differentiable at $x = 0$? Explain.

f will not be differentiable at $x = 0$. Note that, if $x > 0$, then $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(x) \to \infty$ as $x \to 0^+$

If $x < 0$, $f'(x) = \frac{1}{2\sqrt{-x}}$, which also goes to infinity as x approaches 0 (from the left).

37. $f(x) = |\ln(x)|$. Find $f'(x)$.

We can rewrite f (Recall that $\ln(x) < 0$ if $0 < x < 1$)

$$f(x) = \begin{cases} \ln(x), & \text{if } x \geq 1 \\ -\ln(x), & \text{if } 0 < x < 1 \end{cases}$$

and differentiate piecewise:

$$f'(x) = \begin{cases} \frac{1}{x}, & \text{if } x > 1 \\ -\frac{1}{x}, & \text{if } 0 < x < 1 \end{cases}$$

Note that the pieces don't match at $x = 1$; we remove that point from the domain.

38. $f(x) = xe^{g(\sqrt{x})}$. Find $f'(x)$.

$$f'(x) = e^{g(\sqrt{x})} + xe^{g(\sqrt{x})} \cdot g'(\sqrt{x}) \cdot \frac{1}{2}x^{-1/2}$$

39. Find a formula for dy/dx: $x^2 + xy + y^3 = 0$.

$$2x + y + xy' + 3y^2y' = 0 \Rightarrow (x + 3y^2)y' = -(2x + y) \Rightarrow y' = \frac{-(2x + y)}{x + 3y^2}$$

40. Show that 5 is a critical number of $g(x) = 2 + (x - 5)^3$, but that g does not have a local extremum there.

    $g'(x) = 3(x - 5)^2$, so $g'(5) = 0$.

    By looking at the sign of $g'(x)$ (First derivative test), we see that $g'(x)$ is always non-negative, so g does not have a local min or max at $x = 5$.

41. Find the general antiderivative:

    (a) $f(x) = 4 - x^2 + 3e^x$ $F(x) = 4x - \frac{1}{3}x^3 + 3e^x + C$

    (b) $f(x) = \frac{3}{x^2} + \frac{2}{x} + 1$ $F(x) = -3x^{-1} + 2\ln|x| + x + C$

    (c) $f(x) = \frac{1+x}{\sqrt{x}}$

    First rewrite $f(x) = x^{-1/2} + x^{1/2}$, and

$$F(x) = 2x^{1/2} + \frac{2}{3}x^{3/2} + C$$

42. Find the slope of the tangent line to the following at the point (3,4): $x^2 + \sqrt{y}x + y^2 = 31$

$$2x + \frac{1}{2}y^{-1/2}y'x + \sqrt{y} + 2yy' = 0$$

At $x = 3, y = 4$:

$$6 + \frac{3}{4}y' + 2 + 8y' = 0 \quad \Rightarrow \quad y' = \frac{-32}{35}$$

$$y - 4 = \frac{-32}{35}(x - 3)$$

43. Find the critical values: $f(x) = |x^2 - x|$

    One way to approach this problem is to look at it piecewise. Use a table to find where $f(x) = x(x - 1)$ is positive or negative:

$$f(x) = \begin{cases} x^2 - x & \text{if } x \le 0, \text{ or } x \ge 1 \\ -x^2 + x & \text{if } 0 < x < 1 \end{cases}$$

Now compute the derivative:

$$f'(x) = \begin{cases} 2x - 1 & \text{if } x < 0, \text{ or } x > 1 \\ -2x + 1 & \text{if } 0 < x < 1 \end{cases}$$

At $x = 0$, from the left, $f'(x) \to 1$ and from the right, $f'(x) \to -1$, so $f'(x)$ does not exist at $x = 0$.

At $x = 1$, from the left, $f'(x) \to -1$, and from the right, $f'(x) \to 1$, so $f'(x)$ does not exist at $x = 1$.

Finally, $f'(x) = 0$ if $2x - 1 = 0 \Rightarrow x = 1/2$, but $1/2$ is not in that domain. The other part is where $-2x + 1 = 0$, which again is $1/2$, and this time it is in $0 < x < 1$.

The critical points are: $x = 1/2, 0, 1$.

44. Does there exist a function f so that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$ for all x?

SOLUTION: By the MVT

$$f'(x) = \frac{f(2) - f(0)}{2 - 0} = \frac{5}{2}$$

Since $\frac{5}{2} > 2$, there can exist no function like that (that is continuous).

45. Find a function f so that $f'(x) = x^3$ and $x + y = 0$ is tangent to the graph of f.

SOLUTION: If $f'(x) = x^3$, then $f(x) = \frac{1}{4}x^4 + C$. We are also told that $y = -x$ is the equation of a tangent line to the graph of f. Therefore, the slope of the tangent line (at some value of x) is -1:

$$x^3 = -1 \Rightarrow x = -1$$

For our function to have a tangent line with slope -1, we MUST be taking the tangent line at $x = -1$.

Now, what should the value of C be in $f(x)$? We want the graph of $f(x)$ to touch our tangent line at $x = -1$, so the $y-$value of f should be equal to the $y-$value of the tangent line at $x = -1$:

$$f(-1) = \frac{1}{4} + C = 1$$

so that $C = \frac{3}{4}$. Final answer: $f(x) = \frac{1}{4}x^4 + \frac{3}{4}$.

46. Find dy if $y = \sqrt{1-x}$ and evaluate dy if $x = 0$ and $dx = 0.02$. Compare your answer to Δy

$$dy = \frac{-1}{2\sqrt{1-x}}\,dx, \quad \Rightarrow dy = \frac{-1}{2\sqrt{1-0}} \cdot 0.02 = -0.01$$

$$\Delta y = \sqrt{1 - 0.02} - \sqrt{1} = -0.01005...$$

47. Fill in the question marks: If f'' is positive on an interval, then f' is INCREASING and f is CONCAVE UP.

48. If $f(x) = x - \cos(x)$, x is in $[0, 2\pi]$, then find the value(s) of x for which

   (a) $f(x)$ is greatest and least.

   Here we are looking for the maximum and minimum- use a table with endpoints and critical points. To find the critical points,

$$f'(x) = 1 + \sin(x) = 0 \Rightarrow \sin(x) = -1 \Rightarrow x = \frac{3\pi}{2}$$

is the only critical point in $[0, 2\pi]$.

Now the table:

x	0	$\frac{3\pi}{2}$	2π
$f(x)$	-1	$\frac{3}{\pi}2 \approx 4.7$	$2\pi - 1 \approx 5.2$

so f is greatest at $x = 2\pi$, least at 0.

(b) $f(x)$ is increasing most rapidly.

Another way to say this: Where's the maximum of $f'(x)$? We've computed $f'(x)$ to be: $1 + \sin(x)$, so take its derivative: $\cos(x) = 0$. So there are two critical points at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. Checking these and the endpoints:

x	0	$\frac{\pi}{2}$	$\frac{3\pi}{2}$	2π
$f'(x)$	1	2	0	1

so f is increasing most rapidly at $x = \frac{\pi}{2}$.

(c) The slopes of the lines tangent to the graph of f are increasing most rapidly.

Another way to say this: Where is $f'(x)$ increasing most rapidly? At the maximum of $f''(x)$. The maximum of $\cos(x)$ in the interval $[0, 2\pi]$ occurs at $x = 0$ and $x = 2\pi$.

49. Show there is *exactly* one root to: $\ln(x) = 3 - x$

First, to use the Intermediate Value Theorem, we'll get a function that we can set to zero: Let $f(x) = \ln(x) - 3 + x$. Then a root to $\ln(x) = 3 - x$ is where $f(x) = 0$.

First, by plugging in numbers, we see that $f(2) < 0$ and $f(3) > 0$. There is at least one solution in the interval $[2, 3]$ by the Intermediate Value Theorem.

Now, is there more than one solution? $f'(x) = \frac{1}{x} + 1$ which is always positive for positive x. This means that, for $x > 0$, $f(x)$ is always increasing. Therefore, if it crosses the $x-$axis (and it does), then f can never cross again.

50. Sketch the graph of a function that satisfies all of the given conditions:

$$f(1) = 5 \qquad f(4) = 2 \qquad f'(1) = f'(4) = 0$$
$$\lim_{x \to 2^+} f(x) = \infty, \quad \lim_{x \to 2^-} f(x) = 3 \qquad f(2) = 4$$

51. If $s^2 t + t^3 = 1$, find $\frac{dt}{ds}$ and $\frac{ds}{dt}$.

SOLUTION: First, treat t as the function, s as the variable:

$$2st + s^2 \frac{dt}{ds} + 3t^2 \frac{dt}{ds} = 0 \Rightarrow \frac{dt}{ds} = \frac{-2st}{s^2 + 3t^2}$$

For s as the function, t as the variable:

$$\frac{ds}{dt} = \frac{-(s^2 + 3t^2)}{2st}$$

which you can either state directly or show.

52. Rewrite the function as a piecewise defined function (which gets rid of the absolute value signs):

$$f(x) = \frac{|3x+2|}{3x+2} \qquad f(x) = \left|\frac{x-2}{(x+1)(x+2)}\right|$$

SOLUTION: For the first one, we see the denominator and numerator will either be equal (so $f(x) = 1$) if $3x + 2 > 0$ or if $x > -2/3$. The function will be $f(x) = -1$ if $x < -2/3$.

For the second function, it might be helpful to do a sign analysis:

$x-2$	$-$	$-$	$-$	$+$
$x+1$	$-$	$-$	$+$	$+$
$x+2$	$-$	$+$	$+$	$+$
	$x<-2$	$-2<x<-1$	$-1<x<2$	$x>2$

From this, we get:

$$f(x) = \begin{cases} (x-2)/((x+1)(x+2)) \text{ if } -2 < x < -1 \text{ or } x \geq 2 \\ -(x-2)/((x+1)(x+2)) \text{ if } x < -2 \text{ or } -1 < x < 2 \end{cases}$$

53. Find all values of c and d so that f is continuous at all real numbers:

$$f(x) = \begin{cases} 2x^2 - 1 & \text{if } x < 0 \\ cx + d & \text{if } 0 \leq x \leq 1 \\ \sqrt{x+3} & \text{if } x > 1 \end{cases}$$

SOLUTION: We must check $x = 0$ and $x = 1$; f will be continuous everywhere else:

- $f(0) = d$
- $\lim_{x\to 0^+} f(x) = d$
- $\lim_{x\to 0^-} f(x) = -1$
- For f to be continuous at $x = 0$, $d = -1$.

- $f(1) = c + d$
- $\lim_{x\to 1^+} f(x) = 2$
- $\lim_{x\to 1^-} f(x) = c + d$
- Since $d = -1$, we get $c = 3$.

54. Find $f(x)$:

(a) $f'(x) = \frac{3x-1}{\sqrt{x}}$, $f(1) = 2$.

SOLUTION: Remember to simplify first so that $f'(x) = 3x^{1/2} - x^{-1/2}$. Then

$$f(x) = 2x^{3/2} - 2x^{1/2} + C$$

Substituting $f(1) = 2$ gives $C = 2$ so that

$$f(x) = 2x^{3/2} - 2x^{1/2} + 2$$

(b) $f''(x) = 3e^x + 5\sin(x),\ f(0) = 1,\ f'(0) = 2.$
SOLUTION: $f(x) = 3e^x - 5\sin(x) + 4x - 2.$

55. A rectangle is to be inscribed between the $x-$axis and the upper part of the graph of $y = 8 - x^2$ (symmetric about the $y-$axis). For example, one such rectangle might have vertices: $(1, 0), (1, 7), (-1, 7), (-1, 0)$ which would have an area of 14. Find the dimensions of the rectangle that will give the largest area.

SOLUTION: The area of the rectangle is $2xy$, with $y = 8 - x^2$, so we want to maximize $A(x) = 2x(8 - x^2) = 16x - 2x^3$, with $0 \le x \le \sqrt{8}$.

Find the critical points: $x = \sqrt{8/3}$ (only the positive root is in our interval), then substitute these into the area:

x	0	$\sqrt{8/3}$	$\sqrt{8}$
$A(x)$	0	17.4	0

Therefore, the dimensions that give the largest area are

$$x = \sqrt{\frac{8}{3}} \qquad y = \frac{16}{3}$$

56. What is the smallest possible area of the triangle that is cut off by the first quadrant and whose hypotenuse is tangent to the curve $y = 4 - x^2$ at some point?

SOLUTION: If we take the unknown point on the curve to be $(a, 4 - a^2)$, then the equation of the tangent line is:

$$y - (4 - a^2) = -2a(x - a)$$

The x, y intercepts are:

$$x\text{-int} = \frac{a^2 + 4}{2a} \qquad y\text{-int} = a^2 + 4$$

so the area we want to minimize is:

$$F(a) = \frac{(a^2 + 4)^2}{2a} \qquad F'(a) = \frac{2(a^2 + 4)(2a)(2a) - (a^2 + 4)^2 2}{4a^2}$$

Set $F'(a) = 0$ and solve for a: We get $a = \sqrt{4/3}$. Now we build a table. We cannot evaluate F at 0, but as $a \to 0$, we see that $F(a) \to \infty$.

a	0	$\sqrt{4/3}$	2
$F(a)$	∞	12.31	16

so the minimum occurs at $a = \sqrt{4/3}$, with an area of 12.31.

57. You're standing with Elvis (the dog) on a straight shoreline, and you throw the stick in the water. Let us label as "A" the point on the shore closest to the stick, and suppose that distance is 7 meters. Suppose that the distance from you to the point A is 10 meters. Suppose that Elvis can run at 3 meters per second, and can swim at 2 meters per second. How far along the shore should Elvis run before going in to swim to the stick, if he wants to minimize the time it takes him to get to the stick?

    SOLUTION: Setting it up as in class, let x be the distance from A at which Elvis goes into the water. Then the distance he is to run is $10 - x$ on land, and he will swim $\sqrt{x^2 + 49}$ meters to the stick. His time is therefore:

$$T(x) = \frac{10 - x}{3} + \frac{\sqrt{x^2 + 49}}{2}, \quad 0 \leq x \leq 10$$

    Finding the critical point,

$$T'(x) = -\frac{1}{3} + \frac{x}{2\sqrt{x^2 + 49}}$$

    Setting the derivative to zero, we get $x = 14/\sqrt{5}$. Building a table, we have:

x	0	$14/\sqrt{5}$	10
T	6.83	5.94	6.10

    so we choose $x = 14/\sqrt{5}$ to minimize the time.

58. A water tank in the shape of an inverted cone with a circular base has a base radius of 2 meters and a height of 4 meters. If water is being pumped into the tank at a rate of 2 cubic meters per minute, find the rate at which the water level is rising when the water is 3 meters deep. ($V = \frac{1}{3}\pi r^2 h$)

    SOLUTION: If h is the height of the water, and r is the radius, the volume of water is a different than what is given for the whole cone. It is the volume of the full cone, then subtract the "empty" cone. If h is the height of the water, then the volume is:

$$V = \frac{1}{3}\pi 2^2 \cdot 4 - \frac{1}{3}\pi r^2 (4 - h)$$

    From similar triangles, we get $r = \frac{1}{2}(4 - h)$, so our formula becomes:

$$V = \frac{16}{3}\pi - \frac{\pi}{12}(4 - h)^3$$

    Now, treat V, h as functions of time:

$$\frac{dV}{dt} = -\frac{\pi}{12} 3(4 - h)^2(-1) = \frac{\pi}{4}(4 - h)^2 \frac{dh}{dt}$$

    With $dV/dt = 2$ and $h = 3$, we get: $dh/dt = 8/\pi$.

59. A ladder 10 feet long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 foot per second, how fast is the angle between the ground and the ladder changing when the bottom of the ladder is 6 feet from the wall?

    SOLUTION: This is slightly different than one we've seen before. In setting things up, you should have a right triangle whose hypotenuse is 10, angle θ in the lower position, and x is the length adjacent. Then, with θ and x functions of t,

$$\cos(\theta) = \frac{x}{10} \quad \Rightarrow \quad -\sin(\theta)\frac{d\theta}{dt} = \frac{1}{10}\frac{dx}{dt}$$

    Substitute $x = 6$ into the triangle to see that the height (or side opposite θ is 8, so that $\sin(\theta) = 8/10$. We also substitute $dx/dt = 1$ to get:

$$\frac{d\theta}{dt} = -\frac{1}{8}$$

60. If a snowball melts so that its surface area decreases at a rate of 1 square centimeter per minute, find the rate at which the diameter decreases when the diameter is 10 cm. (The surface area is $A = 4\pi r^2$)

    SOLUTION: The change in diameter is -20π.

61. A man walks along a straight path at a speed of 4 ft/sec. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

    SOLUTION: Example 5, Section 3.9