Extra Practice Solutions: Trigonometry

1. For this first set of problems, you should be able to use our two triangles ($\pi/6, \pi/3, \pi/2$ and $\pi/4, \pi/4, \pi/2$) and the unit circle.

   Evaluate the following (exactly, without a calculator):

   (a) $\sin(3\pi/4)$
   SOLUTION: By the unit circle, $\sin(3\pi/4) = \sin(\pi/4)$ and this can be read off of a standard triangle- The value is $1/\sqrt{2}$ (or equivalently, $\sqrt{2}/2$).

   (b) $\cos(-5\pi/4)$
   SOLUTION: In the unit circle, $-5\pi/4$ is in quadrant II, and is the same as $3\pi/4$. The cosine in this quadrant is negative, so the answer is:

   $$\cos(-5\pi/4) = -\cos(\pi/4) = -\frac{1}{\sqrt{2}}$$

   (c) $\tan(2\pi/3)$
   SOLUTION: In the unit circle, $2\pi/3$ is in the second quadrant. In this quadrant, the sine is positive and the cosine is negative, so the tangent is negative. Use this together with the 30-60-90 triangle:

   $$\tan(2\pi/3) = -\tan(\pi/3) = -\frac{\sqrt{3}}{1} = -\sqrt{3}$$

   (d) $\sec(7\pi/6)$
   SOLUTION: In the unit circle, we see that $7\pi/6$ is $\pi/6$ into the third quadrant, so sine and cosine is negative. Use this, together with the 30-60-90 triangle:

   $$\sec(7\pi/6) = \frac{1}{\cos(7\pi/6)} = -\frac{1}{\cos(\pi/6)} - \frac{2}{\sqrt{3}}$$

   (e) $\csc(29\pi/6)$
   SOLUTION: The angle is $\pi/6$ shy of $30\pi/6$, which is 5π. On the unit circle, this is equivalent to π. Overall, the angle is the same as $5\pi/6$ on the unit circle. Since this is in quadrant II, the sine is positive. This, together with the 30-60-90 triangle gives us:

   $$csc(29\pi/6) = \frac{1}{\sin(29\pi/6)} = \frac{1}{\sin(\pi/6)} = 2$$

   (f) $\tan(\pi/4)$
   SOLUTION: At $\pi/4$, sine is equal to cosine, to the tangent is 1.

   For the next two questions, we want to use the general ideas:

- For the "regular" trig functions, sine and cosine are periodic with period 2π. The tangent is periodic with period π.
- For $A\cos(\omega t)$ or $A\sin(\omega t)$, the amplitude is A, the period is $2\pi/\omega$ and the frequency is the reciprocal of the period, or $\omega/(2\pi)$. For the tangent, the period of $\tan(\omega t)$ would be π/ω.
- Adding a constant would just shift the graph up/down, and would not change the amplitude or period.

2. What is the amplitude, period and frequency for $f(x) = 1 + 2\cos(3x)$

   SOLUTION: Amplitude is 2, period is $2\pi/3$, frequency is $3/(2\pi)$.

3. What is the period of $f(x) = \tan(\pi x)$? $f(x) = \cos(x/\pi)$?

   SOLUTION: The period of $\tan(\pi x)$ is 1. The period of $\cos(x/\pi)$ is $2\pi^2$

4. Solve for x:

   (a) $2\cos(x) + 1 = 0$
   SOLUTION: First, $\cos(x) = -\frac{1}{2}$. On the unit circle, we can have angles either in quadrant II or quadrant III. By the 30-60-90 triangle, the solution to $\cos(x) = 1/2$ is $x = \pi/3$. Putting this in the second and third quadrants:

   $$x = \frac{2\pi}{3}, \frac{4\pi}{3} \text{ or we can add any multiple of } 2\pi$$

   (b) $3\cot^2(x) = 1$
   SOLUTION: First the algebra to get

   $$\cot(x) = \pm\frac{1}{\sqrt{3}} \quad \Rightarrow \quad \tan(x) = \pm\sqrt{3}$$

   The tangent function is positive in quadrants I and III, and negative in quadrants II and IV. Furthermore, using a 30-60-90 triangle, $x = \pi/3$.
   Therefore, the solutions to:

   $$\tan(x) = \sqrt{3} \quad \Rightarrow \quad x = \frac{\pi}{3}, \frac{4\pi}{3}$$

   And

   $$\tan(x) = -\sqrt{3} \quad \Rightarrow \quad x = \frac{2\pi}{3}, \frac{5\pi}{3}$$

   And we can add any multiple of π to these. In fact, with that simplification, we could say:

   $$\tan(x) = \sqrt{3} \quad \Rightarrow \quad x = \frac{\pi}{3} + k\pi \text{ for } k = 0, \pm 1, \pm 2, ...$$

   and

   $$\tan(x) = -\sqrt{3} \quad \Rightarrow \quad x = \frac{2\pi}{3} + k\pi \text{ for } k = 0, \pm 1, \pm 2, ...$$

(c) $\sin(x) > \cos(x)$

To solve this, we first look to see where $\sin(x) = \cos(x)$. On the unit circle, this is at $\pi/4$ and $5\pi/4$. Also on the unit circle, this set of points correspond to where the y coordinate is larger than the x coordinate.

SOLUTION: $\dfrac{\pi}{4} < x < \dfrac{5\pi}{4}$

5. For the following, recall that, to compute $\sin^{-1}(x)$ means to find an angle θ so that $\sin(\theta) = x$. Normally, this would mean that there are an infinite number of possible angles (because the sine is not one-to-one); in order to make the inverse sine a *function*, we restricted the possible angles to be between $-\pi/2$ and $\pi/2$. Similarly, we defined the other inverse functions. Here we go:

(a) $\sin^{-1}(0)$ SOLUTION: $0 = \sin^{-1}(0)$ either by unit circle or the graph of sine.

(b) $\sin^{-1}(1)$ SOLUTION: $\pi/2$ is the angle (the only one in $[-\pi/2, \pi/2]$).

(c) $\arcsin(1/2)$ SOLUTION: Use a 30-60-90 triangle to see that the angle is $\pi/6$ (and this is the only angle).

(d) $\sin^{-1}(2)$ SOLUTION: The domain of the inverse sine is $[-1, 1]$, so we cannot compute this quantity (it does not exist).

(e) $\tan^{-1}(-\sqrt{3})$

SOLUTION: The angles used in the inverse tangent are almost the same as the inverse sine (delete the endpoints). The tangent is negative in quadrant IV, and using the 30-60-90 triangle we see that the angle is $-\pi/3$.

(f) $\tan^{-1}(0) = 0$

(g) $\tan^{-1}(1) = \pi/4$

(h) $\sec^{-1}(-2)$

SOLUTION: This is the angle whose secant is -2, which means that the cosine of the angle is $-1/2$. Since we typically restrict the domain of the cosine to be $[0, \pi]$ (to get its inverse), we'll do the same with the secant, except for the secant there is a vertical asymptote at $\pi/2$ (since $cos(\pi/2) = 0$).

With that restriction, we need the angle to be in quadrant II. Using a 30-60-90 triangle, we see that the cosine is $1/2$ means the angle is $\pi/3$. Putting this in quadrant II gives us our answer:

$$\sec^{-1}(-2) = \frac{2\pi}{3}$$

(i) $\sec^{-1}(2/\sqrt{3})$

SOLUTION: Similar to the previous reasoning, we look for an angle in $[0, pi/2) \cup (\pi/2, \pi]$ for which $\cos(\theta) = \sqrt{3}2$, which is $\pi/6$.

6. The trick on these is that, while the inverse function undoes the original function:

$$f^{-1}(f(x)) = x \quad \Rightarrow \quad \sin(\sin^{-1}(x)) \text{ and } \sin^{-1}(\sin(x)) = x$$

we have to use the restricted domain and range (so that the function is one-to-one). Same idea applies to the tangent and inverse tangent.

Here we go:

(a) $\sin^{-1}(\sin(\pi))$
SOLUTION: Since π is not in our restricted domain of $[-\pi/2, \pi/2]$, you should note that the answer is NOT π. Rather, $\sin(\pi) = 0$ and $\sin^{-1}(0) = 0$, so the answer is 0.

(b) $\sin(\sin^{-1}(3/5)$
SOLUTION: Since 3/5 is in the restricted domain of the inverse sine (which is $[-1, 1]$), the answer is 3/5 (that is, we apply the rule $\sin(\sin^{-1}(x)) = x$).

(c) $\tan^{-1}(\tan(\pi/4))$
SOLUTION: $\pi/4$

(d) $\tan^{-1}(\tan(\pi)) = 0$ by reasoning similar to the first problem.

7. Simplify the following expressions (using a triangle). Also think about the value(s) of x for which the simplification is valid.

(a) $\tan(\sin^{-1}(x))$
SOLUTION: Let $\theta = \sin^{-1}(x)$, or $\sin(\theta) = \frac{x}{1}$. Build a right triangle with this relationship (opposite θ has length x, hypotenuse has length 1). Then the length of the side adjacent is $\sqrt{1 - x^2}$ by the Pythagorean Theorem. Finally, we compute the tan:

$$\tan(\theta) = \tan(\sin^{-1}(x)) = \frac{x}{\sqrt{1 - x^2}}$$

(b) $\cos(\tan^{-1}(x))$
SOLUTION: Let $\theta = \tan^{-1}(x)$, or $\tan(\theta) = \frac{x}{1}$. Build a right triangle with this relationship (opposite θ has length x, adjacent has length 1). Then the length of the hypotenuse is $\sqrt{1 + x^2}$ by the Pythagorean Theorem. Now compute the cosine:

$$\cos(\theta) = \cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1 + x^2}}$$

(c) $\sec(\sin^{-1}(x))$
SOLUTION: This triangle is the same as in part (a)- From that:

$$\cos(\theta) = \frac{1}{\sqrt{1 - x^2}} \quad \Rightarrow \quad \sec(\sin^{-1}(x)) = \sqrt{1 - x^2}$$

(d) $\tan(\sec^{-1}(x))$

SOLUTION: Make this triangle consistent with the relation:

$$\cos(\theta) = \frac{1}{x}$$

so that the length of the side adjacent is 1, the length of the hypotenuse is x, which makes the length of the side opposite $\sqrt{x^2 - 1}$. Therefore,

$$\tan(\sec^{-1}(x)) = \sqrt{x^2 - 1}$$

(e) $\cos(2\sin^{-1}(x))$

SOLUTION: Using the hint, if we let $\theta = \sin^{-1}(x)$, then we need to find $\cos^2(\theta)$ and $\sin^2(\theta)$. Build a triangle consistent with the relation:

$$\sin(\theta) = \frac{x}{1}$$

so that the length of the side adjacent to θ is $\sqrt{1 - x^2}$. Then:

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = \left(\sqrt{1 - x^2}\right)^2 - x^2 = 1 - 2x^2$$

(f) $\sin(2\tan^{-1}(x))$

Using the hint, if $\theta = \tan^{-1}(x)$, the we want to compute $\sin(\theta)$ and $\cos(\theta)$. The triangle has the same lengths as (b), so: $\sin(\theta) = x/\sqrt{1 - x^2}$ and $\cos(\theta) = 1/\sqrt{1 - x^2}$:

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta) = \frac{2x}{1 - x^2}$$