SAMPLE EXAM 1 SOLUTIONS

1. Short answer:

   (a) Finish the definition: The limit as x approaches a of $f(x)$ is L if, for every $\epsilon > 0$, there is a $\delta > 0$ such that:
   SOLUTION: If $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$

   (b) Finish the definition: $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$
   Grading note: The limit is a *crucial* part of this definition. Also, remember to keep the arguments consistent- For example, if you're defining $f'(x)$, don't use $f(a+h)$ in the difference quotient.

   (c) True or False (and give a short reason): If f is continuous at $x = a$, then f is differentiable at $x = a$.
   SOLUTION: False. For example, $f(x) = |x|$ is continuous at $x = 0$, but the function is not differentiable at $x = 0$.

2. Find $f'(x)$ directly from the definition of the derivative (using limits and without using l'Hospital's rule):

$$f(x) = \sqrt{1+x}$$

   SOLUTION: Once we set up the difference quotient, multiply by the conjugate:

$$\lim_{h\to 0} \frac{\sqrt{1+x+h} - \sqrt{1+x}}{h} \cdot \frac{\sqrt{1+x+h} + \sqrt{1+x}}{\sqrt{1+x+h} + \sqrt{1+x}} =$$

$$\lim_{h\to 0} \frac{(1+x+h) - (1+x)}{h(\sqrt{1+x+h} + \sqrt{1+x})} = \lim_{h\to 0} \frac{h}{h(\sqrt{1+x+h} + \sqrt{1+x})} = \frac{1}{2\sqrt{1+x}}$$

3. Derive the formula for the derivative of $y = \sin^{-1}(x)$.

   SOLUTION: First we re-write the function so we can implicitly differentiate it.

$$\sin(y) = x$$

   This corresponds to a right triangle with an angle y, opposite length x, hypotenuse 1, and adjacent side $\sqrt{1-x^2}$ (by the Pythagorean Theorem). Now differentiate and convert back to x:

$$\cos(y)y' = 1 \quad \Rightarrow \quad y' = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$$

4. Find dy/dx (solve for it, if necessary):

   (a) $y = \sin^3(x^2+1) + \tan^{-1}(x)$
   SOLUTION: Note the triple composition, so we'll use the chain rule:

$$\frac{dy}{dx} = 3\sin^2(x^2+1)\cos(x^2+1) \cdot 2x + \frac{1}{x^2+1}$$

(b) $y = 3^{1/x} + \sec(x)$

SOLUTION: $y' = 3^{1/x}\ln(3)\cdot(-x^{-2}) + \sec(x)\tan(x)$

(c) $\sqrt{x+y} = 4xy$

SOLUTION: Remember to use the product rule for the right side of the equation-

$$\frac{1+y'}{2\sqrt{x+y}} = 4y + 4xy'$$

Solving for y', break up the fraction on the left and group y' terms together:

$$y'\left(\frac{1}{2\sqrt{x+y}} - 4x\right) = -\frac{1}{2\sqrt{x+y}} + 4y$$

Let's make these single fractions to make the solution clearer:

$$y'\left(\frac{1-8x\sqrt{x+y}}{2\sqrt{x+y}}\right) = \frac{-1+8y\sqrt{x+y}}{2\sqrt{x+y}} \quad \Rightarrow \quad y' = \frac{-1+8y\sqrt{x+y}}{1-8x\sqrt{x+y}}$$

Grading note: You should end up with a single fraction, and not a compound fraction. You may use terms like $(x+y)^{-1/2}$ in your fraction, although it would be nice to clean up the terms.

5. Find the limit, if it exists (you may use any technique from class):

(a) $\lim_{x\to 0} \frac{1-e^{-2x}}{\sec(x)} = \frac{0}{1} = 0$

(b) $\lim_{x\to 4^+} \frac{x-4}{|x-4|} = 1$ NOTE: You should rewrite first as $\frac{x-4}{|x-4|} = \begin{cases} 1 & \text{if } x > 4 \\ -1 & \text{if } x < 4 \end{cases}$

(c) $\lim_{x\to -\infty} \sqrt{\frac{2x^2-1}{x+8x^2}}$

SOLUTION: Divide numerator and denominator by x, and since $x < 0$, we will use the substitution $x = -\sqrt{x^2}$ to simplify. That is:

$$\lim_{x\to-\infty}\sqrt{\frac{2x^2-1}{x+8x^2}} = \lim_{x\to-\infty}\frac{\sqrt{2x^2-1}/x}{\sqrt{x+8x^2}/x} = \lim_{x\to-\infty}\frac{-\sqrt{\frac{2x^2-1}{x^2}}}{-\sqrt{\frac{x+8x^2}{x^2}}} = \lim_{x\to-\infty}\frac{\sqrt{2-\frac{1}{x^2}}}{\sqrt{\frac{1}{x}+8}} = \frac{1}{2}$$

GRADING notes: Even though in this case it turned out that the negative signs canceled, that doesn't always happen. Also, don't leave your answer as $\sqrt{2/8}$- Go ahead and simplify that to 1/2.

(d) $\lim_{x\to\infty} \sqrt{x^2+x+1} - \sqrt{x^2-x}\cdot\frac{\sqrt{x^2+x+1}+\sqrt{x^2-x}}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{(x^2+x-1)-(x^2-x)}{\sqrt{x^2+x+1}+\sqrt{x^2-x}}$

$= \lim_{x\to\infty}\frac{2x-1}{\sqrt{x^2+x+1}+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{(2x-1)/x}{(\sqrt{x^2+x+1}+\sqrt{x^2-x})/x}$

$\lim_{x\to\infty}\frac{2-\frac{1}{x}}{\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x}}} = \frac{2}{2} = 1$

NOTE on 5: It was just by chance that I selected 4 limits for which you would not use l'Hospital's rule- Typically, there might be some of those mixed in as well.

6. (a) Find the general antiderivative of $f(x) = \frac{1+x}{\sqrt[3]{x}}$
SOLUTION: Simplify first; you do not need to break up the antiderivative at $x = 0$ (this would typically be specified- For example, $x > 0$).

$$f(x) = x^{-1/3} + x^{2/3} \quad \Rightarrow \quad F(x) = \frac{3}{2}x^{2/3} + \frac{3}{5}x^{5/3} + C$$

(b) Find a function f so that $f'(x) = x^3$ and $x + y = 0$ is tangent to the graph of f.
SOLUTION: Since $f'(x) = x^3$, then $f(x) = \frac{1}{4}x^4 + C$. There are several ways of reasoning out the value of C- Here is one way to think of it:
Since $x + y = 0$ is tangent to f at some point $x = a$, then we can compare this to the regular form of the tangent line:

$$y = -x \qquad \text{compared to} \qquad y = f(a) + f'(a)(x - a)$$

We see that $f'(a) = -1$, from which we get:

$$a^3 = -1 \quad \Rightarrow \quad a = -1$$

Filling this in now, we compare:

$$y = -x \qquad \text{compared to} \qquad y = f(-1) - (x + 1) = -x + (f(-1) - 1)$$

Therefore, $f(-1) - 1 = 0$, or $f(-1) = 1$. Putting that into our expression for f, we can finally solve for C:

$$\frac{1}{4}(-1)^4 + C = 1 \quad \Rightarrow \quad C = \frac{3}{4} \quad \Rightarrow \quad f(x) = \frac{1}{4}x^4 + \frac{3}{4}$$

(c) Find the displacement of a particle if the acceleration is $a(t) = t + \sqrt{t}$, with $v(1) = 2$ and $s(1) = 1$.
SOLUTION: Rewrite in exponent form, then antidifferentiate twice:

$$a(t) = t + t^{1/2} \quad \Rightarrow \quad v(t) = \frac{1}{2}t^2 + \frac{2}{3}t^{3/2} + C$$

Solve for the constant before continuing: $2 = \frac{1}{2} + \frac{2}{3} + C$, so $C = \frac{5}{6}$, and

$$s(t) = \frac{1}{6}t^3 + \frac{4}{15}t^{5/2} + \frac{5}{6}t + C$$

Solving for the new constant, $1 = \frac{1}{6} + \frac{4}{15} + \frac{5}{6} + C$, so $C = -4/15$ and

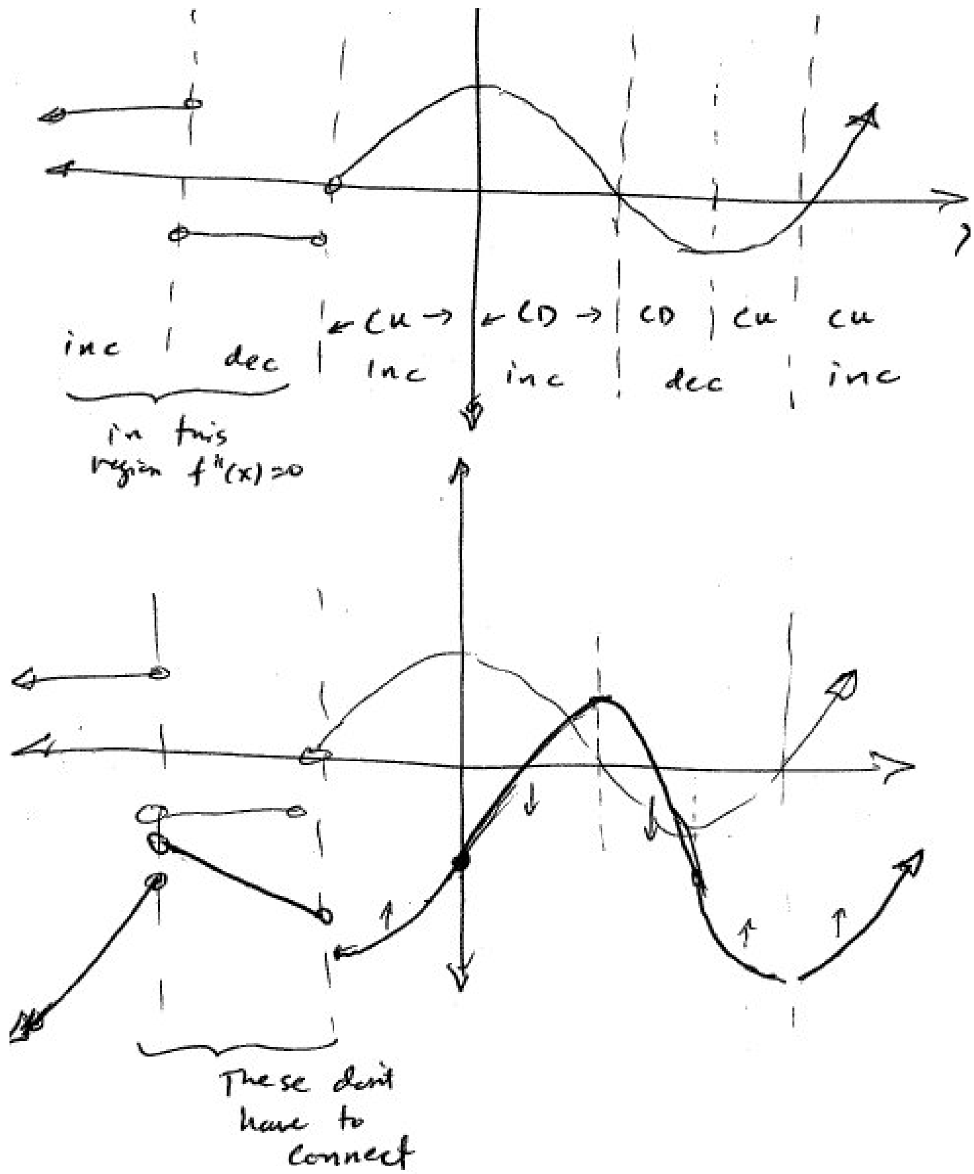
$$s(t) = \frac{1}{6}t^3 + \frac{4}{15}t^{5/2} + \frac{5}{6}t - \frac{4}{15}$$

7. Given the graph of the derivative, $f'(x)$, below, answer the following questions:

(a) Find all intervals on which f is increasing.

(b) Find all intervals on which f is concave up.

(c) Sketch a possible graph of f if we require that $f(0) = -1$.



SOLUTION:

8. A rectangle is to be inscribed between the $x-$axis and the upper part of the graph of $y = 8 - x^2$ (symmetric about the $y-$axis). For example, one such rectangle might have vertices: $(1, 0), (1, 7), (-1, 7), (-1, 0)$ which would have an area of 14. Find the dimensions of the rectangle that will give the largest area.

SOLUTION: Try drawing a picture first: The parabola opens down, goes through the $y-$intercept at 8, and has $x-$intercepts of $\pm\sqrt{8}$.

Now, let x be as usual, so that the full length of the base of the rectangle is $2x$. Then the height is y, or $8 - x^2$. Therefore, the area of the rectangle is:

$$A = 2xy = 2x(8 - x^2) = 16x - 2x^3$$

and $0 \leq x \leq \sqrt{8}$. We see that the area will be zero at the endpoints, so we expect a maximum at the critical point inside the interval:

$$\frac{dA}{dx} = 16 - 6x^2$$

so the critical points are $x = \pm\sqrt{8/3}$, of which only the positive one is in our interval. So the dimensions of the rectangle are as follows (which give the maximum area of approx. 17.4):

$$2x = 2\sqrt{8}3 \qquad y = 8 - \frac{8}{3} = \frac{16}{3}$$

9. A man walks along a straight path at a speed of 4 ft/sec. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

   SOLUTION: The most difficult part of this problem is probably drawing the sketch. Here is a suggestion:

   Draw a triangle whose top is the searchlight and whose base is the wall (the man is standing on the other vertex). Then the length of the leg from the light to the wall is fixed at 20. Define θ to be the acute angle at the searchlight end of the triangle, and x to be the length of the base (against the wall to the man).

   Now we can translate the given information: Given that $dx/dt = 4$, we want to find $d\theta/dt$ when $x = 15$. Here is one relationship between x and θ:

   $$\tan(\theta) = \frac{x}{20} \quad \Rightarrow \quad \sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{20}\frac{dx}{dt} \quad \Rightarrow \quad \frac{d\theta}{dt} = \frac{dx/dt \cdot \cos^2(\theta)}{20}$$

   If $x = 15$, then $\cos(\theta) = \frac{20}{\sqrt{20^2+15^2}} = \frac{4}{5}$, so the answer is:

   $$\frac{d\theta}{dt} = \frac{4 \cdot (4/5)^2}{20} = \frac{16}{125} \approx 0.128$$

   Therefore, the searchlight is rotating at approximately 0.128 radians per second.

10. Estimate the change in the volume of a sphere ($V = \frac{4}{3}\pi r^3$) using differentials, if the *circumference* changes from 2 to 2.1. Give the relative change in volume as well.

    SOLUTION: Use dV to estimate ΔV, but first write V in terms of the circumference $C = 2\pi r$, or $r = C/2\pi$

    $$V = \frac{4}{3}\pi\left(\frac{C}{2\pi}\right)^3 = \frac{1}{6\pi^2}C^3 \quad \Rightarrow \quad dV = \frac{1}{2\pi^2}C^2\,dC\Big|_{C=2,dC=0.1} = \frac{2^2}{2\pi^2}\frac{1}{10} = \frac{1}{5\pi^2} \approx 0.02$$

    For the relative error, for fun we could compute it symbolically first:

    $$\frac{dV}{V} = \frac{C^2 \cdot dC}{2\pi^2} \cdot \frac{6\pi^2}{C^3} = 3\frac{dC}{C} = 3\frac{1/10}{2} = \frac{3}{20} = 0.15$$

    Note that this means the relative error in volume is about 3 times the relative error in the circumference (as approximated by the differentials).

11. Find all values of c and d so that f is continuous at all real numbers:

$$f(x) = \begin{cases} 2x^2 - 1 & \text{if } x < 0 \\ cx + d & \text{if } 0 \le x \le 1 \\ \sqrt{x+3} & \text{if } x > 1 \end{cases}$$

Be sure it is clear from your work that you understand the definition of continuity.

SOLUTION: First, we note that f is continuous for all values of c, d if we remove $x = 0$ and $x = 1$ from the domain, so those are the only points we need to check. We'll check $x = 0$ first:

- $f(0)$ exists? Yes: $f(0) = 0 + d = d$ (Exists for all choices of c, d).
- Does the limit exist at $x = 0$? Check both directions:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2x^2 - 1 = -1 \qquad \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} cx + d = d$$

  For the limit to exist overall, we must have: $d = -1$ (which will also make the limit equal to $f(0)$).

Check $x = 1$ now (and we'll go ahead and replace d with -1):

- $f(1)$ exists? Yes: $f(1) = c - 1$ (Exists for all choices of c, d).
- Does the limit exist at $x = 1$? Check both directions:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} cx - 1 = c - 1 \qquad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \sqrt{x+3} = 2$$

  For the limit to exist overall, we must have: $c - 1 = 2$, or $c = 3$

Therefore, if $c = 3$ and $d = -1$, f will be continuous at every x.