Sample Exam 3

See Sample Exam 1 for the lead-in information.

1. Short answer

   (a) Explain the Mean Value Theorem in terms of velocity:

   Side Note for study: You should know the theorems MVT, IVT and EVT, and give situations in which they are used or perhaps an illustration of each.

   SOLUTION: The Mean Value Theorem says that if the displacement function is continuous and differentiable, and we find the average velocity between times a and b, then at some point in time between a and b, our velocity had to be exactly the average. That is, there is a time c in the time interval $[a, b]$ so that

   $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

   (b) Suppose the line $y = 2x + 1$ is tangent to $f(x)$ at $x = 1$. If $x_1 = 1$ is the initial guess for the root to $f(x)$ using Newton's Method, what is x_2?

   SOLUTION: Newton's Method works by linearizing f at each step, then we substitute the $x-$intercept of the tangent line for the zero of the function. Another way to put this is that the expression:

   $$x_1 - \frac{f(x_1)}{f'(x_1)}$$

   is the $x-$intercept of the line tangent to $f(x)$ at $x = x_1$. In this case, x_2 is the $x-$intercept of the given line:

   $$x_2 = -\frac{1}{2}$$

   (c) True or False, and give a short reason:

   i. $\frac{d}{dx}(10^x) = x10^{x-1}$
   SOLUTION: FALSE. The power rule for derivatives is only valid when the exponent is a constant (and the base is a variable). In this case, the derivative ought to be: $10^x \ln(10)$.

   ii. $\displaystyle\lim_{x\to 1} \frac{x^2 + 6x - 7}{x^2 + 5x - 6} = \frac{\lim_{x\to 1} x^2 + 6x - 7}{\lim_{x\to 1} x^2 + 5x - 6}$
   FALSE: It would be true if the denominator was not zero. Additionally, we see that the limit could be taken by either factoring out $(x-1)$ from the numerator and denominator, or by using l'Hospital's rule (which gives 8/7).

   iii. If $f(x) = x^2$, then the equation of the normal line at $x = 3$ is: $y-9 = \frac{-1}{2x}(x-3)$
   FALSE: It would be true if we EVALUATED the negative reciprocal of the derivative at $x = 3$. That is, the following is the equation of the normal line: $y - 9 = \frac{-1}{6}(x - 3)$

2. Find dy/dx (solve for it if necessary):

(a) $y = 3^{x^2-1} + (x^2-3x+1)^5 + (x^2-1)^{\sin(x)}$

SOLUTION: Take the last term out to do it separately, since we need to do it by logarithmic differentiation:

$$y = (x^2-1)^{\sin(x)} \qquad \Rightarrow \qquad \ln(y) = \sin(x)\ln(x^2-1)$$

so that

$$\frac{y'}{y} = \cos(x)\ln(x^2-1) + \sin(x)\frac{2x}{x^2-1}$$

Simplify:

$$y' = (x^2-1)^{\sin(x)}\left(\cos(x)\ln(x^2-1) + \frac{2x\sin(x)}{x^2-1}\right)$$

Now put everything together:

$$y' = 3^{x^2-1}(2x) + 5(3x^2-3x+1)^4(2x-3) + (x^2-1)^{\sin(x)}\left(\cos(x)\ln(x^2-1) + \frac{2x\sin(x)}{x^2-1}\right)$$

(b) $y = \dfrac{1-2x}{\sqrt[3]{x^5}}$

SOLUTION: Simplify that first! $y = x^{-5/3} - 2x^{-2/3}$, so that

$$y' = -\frac{3}{2}x^{-2/3} - 6x^{1/3} + C$$

We'll assume that $x > 0$, otherwise we would need another constant for $x < 0$.

(c) $x\sin(y) + y\sin(x) = 1$

SOLUTION: Implicit differentiation (and the product rule)

$$\sin(y) + x\cos(y)y' + y'\sin(x) + y\cos(x) = 0$$
$$y'(x\cos(y) + \sin(x)) = -\sin(y) - y\cos(x)$$
$$y' = \frac{-\sin(y) - y\cos(x)}{x\cos(y) + \sin(x)}$$

(d) $y = \ln|\csc(3x) + \cot(3x)|$ This one cries out to be simplified:

$$y' = \frac{1}{\csc(3x) + \cot(3)}(-\csc(3x)\cot(3x)3 - \csc^2(3x)3) = -3\csc(3x)$$

3. Find all vertical and horizontal asymptotes of $f(x) = \dfrac{2x^2-2}{x^2-x-2}$

SOLUTION: For the vertical asymptotes, it is clearer if we factor first:

$$\frac{2x^2-2}{x^2-x-2} = \frac{2(x-1)(x+1)}{(x+1)(x-2)} = \frac{2(x-1)}{(x-2)} \quad \text{for } x \neq -1$$

Therefore, there is a hole in the graph at $x = -1$, and a vertical asymptote at $x = 2$. To find the horizontal asymptote, take the limit out to $\pm\infty$, which is easy to compute via l'Hospital's rule to be 2 (therefore, the horizontal asymptote is $y = 2$).

4. Derive the formula for the derivative of $y = \sec^{-1}(x)$:

   SOLUTION: First re-write this as $\sec(y) = x$. This defines the three lengths of a right triangle where one of the acute angles is y, the length of the hypotenuse is x, and the length of the side adjacent is 1. Therefore, the length of the third side (the opposite) is $\sqrt{x^2-1}$. Now differentiate the expression implicitly, and back-substitute using the triangle:

$$\sec(y)\tan(y)y' = 1 \quad \Rightarrow \quad y' = \frac{1}{\sec(y)\tan(y)} = \frac{1}{x\sqrt{x^2-1}}$$

5. Find $f'(1)$ using the definition of the derivative (using limits and you may not use l'Hospital's rule), if $f(x) = \dfrac{x}{x+1}$

   NOTE: Often it is easier to compute the derivative at a specific number if you evaluate at the number. You'll see it below.

   Use the definition, then get a common denominator (and simplify):

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{\frac{1+h}{2+h} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{1}{h}\left(\frac{2(1+h)-(2+h)}{2(2+h)}\right) = \frac{1}{4}$$

6. Find the limit, if it exists (you may use any method from class):

   (a) $\displaystyle\lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2}$

   SOLUTION: Multiply by the conjugate and factor the denominator:

$$\lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2} \cdot \frac{\sqrt{x+6}+x}{\sqrt{x+6}+x} = \lim_{x\to 3} \frac{-x^2+x+6}{x^2(x-3)(\sqrt{x+6}+x)} =$$

$$\lim_{x\to 3} \frac{-(x-3)(x+2)}{x^2(x-3)(\sqrt{x+6}+x)} = \lim_{x\to 3} \frac{-(x+2)}{x^2(\sqrt{x+6}+x)} = -\frac{5}{54}$$

   (b) $\displaystyle\lim_{x\to -\infty} \frac{\sqrt{x^2-9}}{2x-6}$

   SOLUTION: Divide by x, then remember to substitute $x = -\sqrt{x^2}$ since $x < 0$:

$$\lim_{x\to -\infty} \frac{\sqrt{x^2-9}/x}{(2x-6)/x} = \lim_{x\to -\infty} \frac{-\sqrt{\frac{x^2-9}{x^2}}}{2-\frac{6}{x}} = \lim_{x\to -\infty} \frac{-\sqrt{1-\frac{9}{x^2}}}{2-\frac{6}{x}} = -\frac{1}{2}$$

   (c) $\displaystyle\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{e^x-1}\right)$

   SOLUTION: Combine into a single fraction first, then see if l'Hospital's rule will work

$$\lim_{x\to 0^+} \frac{\mathrm{e}^x-1-x}{x(\mathrm{e}^x-1)} = \lim_{x\to 0^+} \frac{\mathrm{e}^x-1}{(\mathrm{e}^x-1)+x\mathrm{e}^x} = \lim_{x\to 0^+} \frac{\mathrm{e}^x}{\mathrm{e}^x+\mathrm{e}^x+x\mathrm{e}^x} = \frac{1}{2}$$

(d) $\lim_{x\to 0}(1-2x)^{1/x}$
SOLUTION: This was the last type of expression we looked at in the section on l'Hospital's rule. Take the logarithm, then take the limit, then exponentiate back to get the overall limit:

$$\lim_{x\to 0}\ln((1-2x)^{1/x}) = \lim_{x\to 0}\frac{\ln(1-2x)}{x} = \lim_{x\to 0}\frac{-2/(1-2x)}{1} = -2$$

so the limit overall is e^{-2}.

7. (a) A company estimates that the marginal cost (in dollars per item) of x items is $1.92 - 0.002x$.

   i. What is the marginal cost of 200 items, and how should we interpret that (that is, what does the marginal cost represent)?
   SOLUTION: The marginal cost of 200 items represents (approximately) the cost to produce one more unit (or the 201st unit). Another way to state this is:

   $$C'(200) \approx C(201) - C(200)$$

   And in this case, $C'(200) = \$1.52$

   ii. If the cost of producing one item is \$562, find the cost of producing 100 items.
   SOLUTION: The cost function is the antiderivative of the marginal cost, so using the constant K:

   $$C(x) = 1.92x + 0.001x^2 + K$$

   And we are given that $C(1) = 562$, so: $K = 560.081$, or

   $$C(x) = 560.08 + 192x + 0.001x^2 \quad \Rightarrow \quad C(100) = \$742.08$$

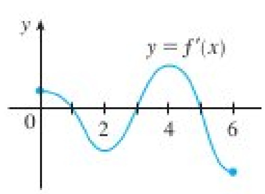
   (b) Find f if $f''(x) = 6x + \sin(x)$ if $f'(0) = 0$ and $f(0) = 3$.
   SOLUTION: Antidifferentiate twice, using the information provided to solve for the arbitrary constants:

   $$f'(x) = 3x^2 - \cos(x) + C_1 \quad \Rightarrow \quad f'(x) = 3x^2 - \cos(x) + 1$$

   $$f(x) = x^3 - \sin(x) + x + C_2 \quad \Rightarrow \quad f(x) = x^3 - \sin(x) + x + 3$$

8. Exercise 53, Section 4.9, p. 349. SOLUTION: See text.

9. The following is the graph of $f'(x)$:



(From exercise 6, section 4.3)

   (a) On what intervals is f increasing or decreasing?
   SOLUTION: The function f is increasing where the graph of $f'(x)$ is above the $x-$axis: $(0,1), (3,5)$.
   The function f is decreasing where the graph of $f'(x)$ is below the $x-$axis: $(1,3), (5,6)$

(b) On what intervals is f concave up or concave down?

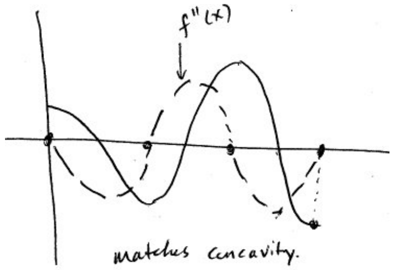
SOLUTION: The function f is concave up where the graph of $f'(x)$ is increasing: $(2, 4)$.

The function f is concave down where the graph of $f'(x)$ is decreasing: $(0, 2)$ and $(4, 5)$.

(c) At what points does f have a local maximum?

SOLUTION: f has a local max that occurs at both $x = 1$ and $x = 5$ (the derivative goes from positive to negative- That is the first derivative test).

(d) Sketch a graph of f''.



10. What is the smallest possible area of the triangle that is cut off by the first quadrant and whose hypotenuse is tangent to the curve $y = 4 - x^2$ at some point?

SOLUTION: First, if $f(x)$ is the function, then the tangent line at some point $x = a$ is: $y - f(a) = f'(a)(x - a)$. If $a > 0$ and $f'(a) < 0$, then the line cuts off a triangle in the first quadrant. The area of the triangle is 1/2 times the product of the $x-$ and $y-$intercepts.

The $y-$intercept of the tangent line is found by setting $x = 0$ and solving for y, and we recall the $x-$intercept from Newton's Method (or you can re-solve for it):

$$y = f(a) - af'(a) \qquad x = a - \frac{f(a)}{f'(a)} \qquad A = \frac{1}{2}xy$$

Now substitute $f(x) = 4 - x^2$ and we get the following (I used a product rule, but you could also use a quotient rule):

$$A = \frac{(a^2+4)^2}{4a} \qquad \Rightarrow \qquad \frac{dA}{da} = a^2 + 4 - \frac{(a^2+4)^2}{4a^2}$$

Factor the derivative, then set it to zero and solve:

$$\frac{dA}{da} = \frac{(a^2+4)}{4a^2}(4a^2 - (a^2+4)) = \frac{(a^2+4)(3a^2-4)}{4a^2} = 0$$

Therefore, $a = 2/\sqrt{3}$, which gives an area of $32/(3\sqrt{3})$, or approx 6.158.

11. The period of oscillation of a pendulum is $P = 2\pi\sqrt{\frac{L}{32}}$, where L is the length of the pendulum. Estimate the change in P using differentials, if the length is changed from 2 to 2.1.

SOLUTION: First, pull the constant all the way out:

$$P = \frac{2\pi}{\sqrt{32}}L^{1/2} = \frac{\pi}{2\sqrt{2}}L^{1/2}$$

Now using differentials:

$$dP = \frac{\pi}{4\sqrt{2}} L^{-1/2}\, dL = \frac{\pi}{4\sqrt{2}\sqrt{2}} \frac{1}{10} = \frac{\pi}{80} \approx 0.0393$$

12. It has been estimated that since the second half of the 19th century, the population of the United States doubles approximately every 56 years. If the current population is approximately 311 million, when will the population reach half a billion?

SOLUTION: The first piece of information can be used to find the rate of growth

$$2P = P\mathrm{e}^{56\,r} \quad \Rightarrow \quad r = \frac{\ln(56)}{2} \approx 2.0127$$

Now, how long to reach 500 million?

$$500 = 311\mathrm{e}^{rt} \quad \Rightarrow \quad t = \frac{1}{r}\ln\left(\frac{500}{311}\right) = \frac{56\ln(500/311)}{\ln(2)} \approx 38.36 \text{ years}$$

Side Note for Study: Some biologists like to use the model $y(t) = P_0 2^{kt}$ instead if $y(t) = P_0\mathrm{e}^{rt}$. Starting from the biologists' model, is there a formula that relates k and r?

SOLUTION TO side note: Since, for any number $A > 0$, we can express it as: $A = \mathrm{e}^{\ln(A)}$, then

$$2^{kt} = \left(\mathrm{e}^{\ln(2)}\right)^{kt} = \mathrm{e}^{\ln(2)\,kt}$$

so $r = k\ln(2)$, and the two functions are actually the same (we like base e because of the derivative formulas are easy to work with).